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Mathematical Reviews

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FOUNDATIONS

Barcan, Ruth C. The identity of individuals in a strict functional calculus of second order. J. Symbolic Logic 12, 12–15 (1947).

In this paper the author develops a theory of identity among individuals for each of two second order functional calculi based on strict implication. These two second order calculi, $S2^2$ and $S4^2$, the author forms from her two strict functional calculi of first order, $S2^1$ and $S4^1$, respectively [see the same J. 11, 1–16, 115–118 (1946); these Rev. 8, 125, 306], by adding propositional variables, variables whose values are functions of individuals, an abstraction operator and appropriate axioms.

The author next defines two relations of identity, one strict and one material. Two individuals x and y are strictly identical (symbolically xIy) just in case the membership of x in any class z strictly implies that y is a member of z. Analogously, x and y are materially identical (symbolically xI_my) just in case the implication of membership in z is material. For $S2^2$ (and a fortiori for the stronger $S4^2$) the author proves such expected properties of I and I_m as reflexivity, symmetry, transitivity and the strict equivalence of 'xIy' to 'it is necessary that xI_my.' But, although demonstrable in $S4^2$, the expected strict equivalence of '$(\exists x)(xIy\cdot x\epsilon A)$' to '$y\epsilon A$' cannot be proved in $S2^2$. The two systems also differ in that I and I_m can be demonstrated materially equivalent in $S2^2$, strictly equivalent in $S4^2$.

G. D. W. Berry (Princeton, N. J.).

Shestakov, V. Representation of characteristic functions of propositions by expressions realizable by relay-contact circuits. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 529–554 (1946). (Russian. English summary)

The function $\omega_\alpha(p)$ of the proposition p is defined by $\omega_\alpha(p)=\omega$ if p is true and $\omega_\alpha(p)=\alpha$ if p is false; $\infty_0(p)=[p]$. Then $[\sim p]=[p]^{-1}$; $[p\vee q]=[p]+[q]$; $[p\cdot q]=[p]*[q]$, where $x*y=(x^{-1}+y^{-1})^{-1}$. On the other hand, let Cx designate the conductivity of an electric contact, so that $Cx=\infty$ when x is shut and $Cx=0$ when x is open. Then $(Cx)^{-1}=Cy$ if y is a contact which is open when x is shut and inversely; $Cx+Cy$ is the conductivity of x and y in series; $Cx*Cy$ that of x and y shunted. Thus to every expression f from the calculus of propositions, built up with the variables $p_1, \cdots, p_n$ by means of negation, conjunction and disjunction, an electric circuit R, consisting of contacts and relays, may be constructed, such that (1) to every p_i corresponds a contact x_i in R; (2) R contains a contact z such that, for every system of truth-values $(\infty, 0)$ of the p_i, if the x_i are so placed that $Cx_i=[p_i]$ $(i=1, \cdots, n)$, then $Cz=[f]$. By the relation $\omega_\alpha(p)=\omega*[p]+\alpha*[p]^{-1}$ this result is extended to the functions $\omega_\alpha(p)$. A further extension to n-valued logic is obtained by using n-position switches. *A. Heyting.*

Carnap, Rudolf. Modalities and quantification. J. Symbolic Logic 11, 33–64 (1946).

This paper gives a survey of results concerning logical systems which contain a notion of logical necessity. A propositional algebra of this type was introduced by C. I. Lewis [see Lewis and Langford, Symbolic Logic, Century Co., New York, 1932; the original papers appeared from 1913 onwards]. Carnap discusses the modal systems on the basis of his recent works ["Introduction to Semantics," "Formalization of Logic," Harvard University Press, 1942, 1943; these Rev. 4, 209]. In this approach the subject of discussion is always a symbolic system or "language" which is then studied by means of another language, the "metalanguage." Two sorts of systems are considered, the syntactical and the semantical. In the former the system is defined by axioms and rules of procedure and no reference need be made to the meanings of the symbols. In the semantical system the rules specify a mapping of the sentences of the system onto a Boolean algebra (or generalization thereof), the element associated with a sentence being called its L-range. Such a system is called semantical because these rules are tied up with references to meaning but to the reviewer this seems unessential: indeed, one can abstract from meanings just as easily in the one kind of system as in the other.

From the mathematical point of view the essential feature of a semantical system is its genetic character: the L-range of a compound sentence is determined in terms of its structure as a function of the L-ranges of its constituents. If the Boolean algebra has two elements, we have the "truth-table method" of, for example, the algebra of propositions. If there are more than two L-ranges, then a sentence can be defined to be L-true if its L-range is universal and to be true (without qualification) if it contains a particular atomic L-range called the real state. Carnap aims to define his necessity operator $N\cdots$ so that Np is true if and only if p is L-true. By considering the meaning of this notion the author argues that the only acceptable way to do this is to assign Np the universal range when p is L-true and the null range otherwise. This leads to the form of strict implication called by Lewis and Langford S5 and studied by M. Wajsberg [Monatsh. Math. Phys. 40, 113–126 (1933)]. Carnap formulates this, first as a semantical system, then as a syntactical one, and discusses the relations between the two formulations. He gives a variant of Wajsberg's reduction to normal form and proof of completeness. He then formulates in the same way a form of the first order predicate calculus with equality. The reduction process and completeness proofs in this case are naturally partial.

The reviewer finds the treatment excessively condensed. He believes he has detected one error, viz., in T6–1f on p. 45 it seems to be alleged, for instance, that if $\sim Np\vee\sim Nq$ is L-true, then $N(\sim p\vee\sim q)$ is L-true; if so, then a counterexample is the case where p and q are true but not L-true. The treatment is also frankly nonfinitary and the author pays no attention to the doubts which some mathematicians have as to the meaning of falsity and negation.

H. B. Curry (State College, Pa.).

★Carnap, Rudolf. Meaning and Necessity. A Study in Semantics and Modal Logic. The University of Chicago Press, Chicago, Ill., 1947. viii+210 pp. $5.00.

This book describes a semantical method called the method of extension and intension, which may be used as the basis for a modal logic with quantified variables. The discussion is partly formalized and is clear and precise. It is maintained that the distinction between the extension and the intension of symbolic or verbal expressions is in their interpretations only; the expressions themselves should be regarded as neutral. Thus classes and properties are respectively the extensions and intensions of neutral expressions called predicators. Similarly sentences are neutral expressions whose extensions are truth-values and whose intensions are propositions. Likewise expressions for individuals have a dual interpretation; their extensions are individuals, while their intensions are individual concepts.

The intensions of expressions are derived from the semantical notions of *L*-true and *L*-equivalent (meaning logically true and logically equivalent). A sentence is *L*-true if its truth follows from semantical rules alone without reference to extralinguistic facts. If a sentence is *L*-true it may be regarded as necessarily true. This notion of necessity leads to modal logics. The object languages considered are symbolic, but for clarity the author takes his metalanguages to be part of English. It is shown that expressions in the metalanguage may also be regarded as neutral, having both extensions and intensions. The relation of the present theory of intensional systems to others, especially those of Frege, Russell, Lewis, Church and Quine is examined thoroughly. In particular, it is pointed out that the usual difficulties connected with the name-relation are avoided. A criticism of the present theory by Quine is presented, together with the author's reaction to this criticism.

There are five chapters entitled: (I) The method of extension and intension, (II) *L*-determinacy, (III) The method of the name-relation, (IV) On metalanguages for semantics, (V) On the logic of modalities. The last chapter is an introduction to systems of modal logic with quantifiers. However, it is not a complete theory of such systems. A sketch of such a system has been published elsewhere by the author [see the preceding review]. A complete theory of modal logics is reserved for a later book. *O. Frink.*

Toledo Toledo, Raimundo. Mathematical foundations of a structural logic. Euclides, Madrid **6**, 554–560, 614–620 (1946). (Spanish)

The author proposes to give a new foundation of logic, less dependent on grammar than the traditional approach. In the first part he studies the inclusion between classes and introduces many new notations. In the second part rules of the following type are deduced: if A and B are identical classes, then also their complementary classes are identical. *J. J. de Iongh* (Amsterdam).

Destouches-Février, Paulette. Sur la notion d'adéquation et le calcul minimal de Johansson. C. R. Acad. Sci. Paris **224**, 545–547 (1947).

This note deals with nonclassical logics appropriate to physical theories. A proposition of such a theory is called adequate relative to a body of experimental facts if it is consistent with them. This notion of adequation leads to logics weaker than classical logic. It is claimed that the minimal calculus of Johansson is the weakest suitable logic of this kind. Adequation may also be treated as a modality, as in modal logics. *O. Frink* (State College, Pa.).

Destouches-Février, Paulette. Adéquation et développement dialectique des théories physiques. C. R. Acad. Sci. Paris **224**, 803–805 (1947).

This note contains further details of the author's calculus of adequation [see the preceding review]. Given a set of experimental facts, she describes a process for constructing a physical theory based on a calculus of adequation as logic, the theory being perfectly adequate relative to the facts in a sense which is defined. The process of construction is called dialectic (after Hegel) and suggests a general theory called the dialectic calculus. *O. Frink.*

Bouligand, Georges. Sur une catégorie de propositions. C. R. Acad. Sci. Paris **223**, 495–496 (1946).

The family of "passages" from one object of a class K to another contains, with the passage $O_1 \to O_2$, the passage $O_2 \to O_1$ and, with the two passages $O_1 \to O_2$, $O_2 \to O_3$, the passage $O_1 \to O_3$. An "operation" furnishes a one-to-one representation of the class K on itself and is determined without ambiguity by a pair of elements O_1 and O_2, O_1 being arbitrary and O_2 the image of O_1. If there is a group of operations, then one can associate a group with each proposition P whose statement makes sense for the objects of the class K. Two illustrations of propositions which do not fall into this category are given. *A. Schwartz.*

Bouligand, G. Les nouveaux problèmes de la formation mathématique. Rev. Gén. Sci. Pures Appl. **53**, 183–186 (1946).

Brouwer, L. E. J. Directions of intuitionistic mathematics. Nederl. Akad. Wetensch., Proc. **50**, 339 = Indagationes Math. **9**, 197 (1947). (Dutch)

Muller-Oikonomos, Sof. The three basic directions in the foundations of mathematics. Bull. Soc. Math. Grèce **21**, 67–103 (1941). (Greek)

A descriptive sketch of the doctrines and methods of Russell, Brouwer and Hilbert.

Black, Max. Professor Broad on the limit theorems of probability. Mind **56**, 148–150 (1947).

Cf. Mind **53**, 1–24, 97–119, 193–214 (1944); these Rev. **6**, 32.

Taylor, Angus E. Some aspects of mathematical research. American Scientist **35**, 211–223 (1947).

ALGEBRA

Good, I. J. Normal recurring decimals. J. London Math. Soc. **21**, 167–169 (1946).

The author constructs a circular array of a^r 0's, 1's, $\cdots$, $(a-1)$'s such that all the a^r combinations of 0's, 1's, $\cdots$, $(a-1)$'s taken r at a time occur once and only once. In the special case when $a=2$, de Bruijn [Nederl. Akad. Wetensch., Proc. **49**, 758–764 = Indagationes Math. **8**, 461–467 (1946); these Rev. **8**, 247] also solved this problem; he also proved that the number of such arrays is $2^{f(n)}$, $f(n)=2^{n-1}-n$. *P. Erdös* (Syracuse, N. Y.).

Rees, D. Note on a paper by I. J. Good. J. London Math. Soc. 21, 169–172 (1946).

The author gives a different proof of the theorem stated in the preceding review. *P. Erdös* (Syracuse, N. Y.).

Easterfield, Thomas E. A combinatorial algorithm. J. London Math. Soc. 21, 219–226 (1946).

The algorithm is a process for determining all minimum sums of n elements taken from an n^2 square array of positive integral numbers, no two from the same row or column. A sub-array of r columns is said to be adjusted if each set of s columns has row-minima from at least s rows, exactly adjusted if the r columns have row-minima from exactly r rows; the process consists of adjusting the array, column by column from the left. *J. Riordan* (New York, N. Y.).

Petr, K. Ein Satz über die Koeffizienten symmetrischer Funktionen. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 43, 132–143 (1942).

The theorem in question relates the coefficients in the expansion of a monomial symmetrical function, say $S_0=[1^{p_1}2^{p_2}\cdots s^{p_s}]$, in terms of elementary functions $a_k=(-)^k(1^k)$ to corresponding coefficients in the expansions of its Hammond derivates: $S_k=[1^{p_1}\cdots k^{p_k-1}\cdots s^{p_s}]$. If a term in the expansion of S_0 is

$$A^{(0)}a_{q_0}^{r_0}a_{q_1}^{r_1}\cdots a_{q_t}^{r_t},\quad \sum q_i r_i=\sum ip_i=w,$$

and $q_0\geqq s$, the corresponding term in the expansion of S_k is $A^{(k)}a_{q_0-k}a_{q_0}^{r_0-1}\cdots a_{q_t}^{r_t}$ and the identity states that $\sum_0^s I_k A^{(k)}=0$, with I_k counting any repetitions of a_{q_0-k} which appear ($I_0=r_0$). The proof is by Hammond's operators, the fundamental properties of which the author re-derives. The theorem provides a neat and quick means of finding expansions of monomial functions from those of lower weight.

J. Riordan (New York, N. Y.).

Petr, K. Eine Identität aus der Theorie der symmetrischen Funktionen und ihre Anwendung. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 43, 144–162 (1942).

The identity proved is in form identical with the known expression of monomial symmetric functions in terms of power sums but has an extended meaning: the monomial function of n parts becomes a sum over n sets of variables, each set having N elements, and the power sum with index a sum of s parts of the monomial function becomes a sum over a product of s variables, the kth term containing the kth variable from each of the sets indicated by the parts. It is used to generalize Newton's equations and the relations of Hammond operators to symmetric functions of several systems of variables, reaching known results [cf. P. A. MacMahon, Combinatory Analysis, vol. 2, Cambridge University Press, 1916, pp. 280 et seq.]. *J. Riordan.*

Aubert, Karl E. Summation of some series of binomial coefficients by means of Cauchy's integral formula. Norske Vid. Selsk. Forh., Trondhjem 17, no. 21, 86–88 (1944).

Using Cauchy's formula for the derivatives of an analytic function the author proves the well-known formula

$$\sum_{s=0}^{n}\binom{k+s}{k}=\sum_{s=0}^{n}\binom{k+s}{s}.$$

By the same method he obtains the sums of more general series of the form $\sum_{s=0}^{n}f(s)\binom{k+s}{k}$, in particular, for $f(s)=s$ and $f(s)=s^2$. *A. Brauer* (Chapel Hill, N. C.).

Duparc, H. J. A. On some determinants. Nederl. Akad. Wetensch., Proc. 50, 157–165 = Indagationes Math. 9, 120–128 (1947).

The author establishes three theorems concerning the evaluation of special determinants and gives applications of the results. Let $|e_{rs}|$ denote the determinant with element e_{rs} in the rth row and sth column. First theorem: if $x_r+y_s\neq0$ ($r, s=1, \cdots, n$), then

$$\left|\frac{x_r+z_s}{x_r+y_s}\right|=\frac{XY\prod_{s=1}^{n}(z_s-y_s)}{\prod_{r,s=1}^{n}(x_r+y_s)}\left\{1-\sum_{r=1}^{n}\frac{(y_r+x_1)\cdots(y_r+x_n)}{(y_r-z_r)q'(y_r)}\right\},$$

where

$$X=\prod_{1\leqq s<r\leqq n}(x_r-x_s),\quad Y=\prod_{1\leqq s<r\leqq n}(y_r-y_s),\quad q(y)=\prod_{r=1}^{n}(y-y_r).$$

In the second theorem, the author shows that the sum inside the braces can be evaluated in terms of the residues of a simple function when $y_s\neq z_s$ for $s=1, \cdots, n$. The theorems are used to evaluate $|\cot(a_r+b_s)|$, $|\tan(a_r+b_s)\tan(a_r-b_s)|$, $|\sin(a_r+b_s)/\sin(a_r-b_s)|$ and numerous other similar determinants. *G. B. Price* (Lawrence, Kan.).

van der Woude, W. On parametric representations with application to Cayley's formula for the representation of the orthogonal determinant. Nederl. Akad. Wetensch., Proc. 49, 866–877 = Indagationes Math. 8, 537–548 (1946). (Dutch)

The relations $x_1=t_5t_2$, $x_2=t_1t_3$, $x_3=t_2t_4$, $x_4=t_3t_5$, $x_5=t_4t_1$ ($\sum t_i=0$) give a parametric representation of Segre's surface $x_1x_2x_3+x_2x_3x_4+x_3x_4x_5+x_4x_5x_1+x_5x_1x_2=0$ in R_4. The author draws attention to the paradoxical fact that the section of the surface with $x_1=0$ seems to be $t_5=0$, $t_2=0$, that is, $x_1=x_4=0$, $x_1=x_3=0$; obviously it also contains the plane $x_1=x_2+x_5$. Other examples are given and the lacking parts of the section found by a limiting process. The theory is applied to orthogonal matrices $\|a_{ik}\|$, with $|a_{ik}|=1$. If $a'_{ii}=a_{ii}+1$, $a'_{ik}=a_{ik}$ ($i\neq k$) and b_{ik} is the cofactor of a'_{ik}, then $b_{ii}=\frac{1}{2}|a'_{ik}|$, $b_{ik}=-b_{ki}$ ($i\neq k$) and if c_{ik} is the cofactor of b_{ik}, then Cayley's formulae $a_{rr}=2b_{11}c_{rr}/|b_{ik}|-1$, $a_{rs}=2b_{11}c_{rs}/|b_{ik}|$ ($r\neq s$) give a representation of the orthogonal matrices by means of the parameters b_{ik}. But if $|b_{ik}|=0$, that is, $|a'_{ik}|=0$, we have exceptional cases. The author studies them especially for $n=2, 3, 4$. *O. Bottema* (Delft).

***Facciotti, G. Sopra una trasformazione nei determinanti di 2° ordine e su un rettangolo numerico ad essa connesso.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 1005–1012. Edizioni Cremonense, Rome, 1942.

Let

$$\Delta=\begin{vmatrix}a_1 & a_2\\ a_4 & a_3\end{vmatrix}$$

and consider the transformations

$$\Phi_s(\Delta)=\Gamma_s=\begin{vmatrix}a_1+a_2 & a_2+a_3\\ a_4+a_1 & a_3+a_4\end{vmatrix},\quad \Phi_d(\Delta)=\Gamma_d=\begin{vmatrix}a_1-a_2 & a_2-a_3\\ a_4-a_1 & a_3-a_4\end{vmatrix}.$$

The author shows that $\Gamma_s{}^n=(-4)^p\Gamma_s{}^q$, $\Gamma_d{}^n=(-4)^p\Gamma_d{}^q$, where p and q are positive integers satisfying the relation $n=2p+q$ (with $q=1$ or 2 according as n is odd or even) and the superscripts indicate iteration of the transformations. The point P of three-space whose homogeneous coordinates are the elements of Δ is transformed by T: $x_1'=x_1+x_2$, $x_2'=x_2+x_3$, $x_3'=x_3+x_4$, $x_4'=x_1+x_4$ into the point whose coordinates are the elements of Γ_s. This singular transformation T sends the

three-space into the plane π: $x_1'-x_2'+x_3'-x_4'=0$, while iterates of T map π onto itself. Similar remarks apply to Γ_4. In the second part of the paper the author considers a table of four columns where the entries in the 4 columns of the jth row are the coefficients of a_1, a_2, a_3, a_4 respectively of the element in the first row and column of the jth transform of Δ by Φ_s (the other elements of this transform are obtained from this one by cyclic permutation of the indices 1, 2, 3, 4). A formula is derived for calculating directly an element of assigned position in this table. *L. M. Blumenthal.*

Cheng, Tseng-Tung. Generalisation of De Moivre's and Fourier's theorems to matrices. Coll. Papers Sci. Engin. Nat. Univ. Amoy 1, 65–68 (1943).

This paper generalizes a theorem given by A. Pen Tung Sah [Elec. Engrg. **60**, 615–616 (1941)], which is equivalent to the well-known fact that certain matrices like

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x & iy \\ iy & x \end{pmatrix}$$

behave like the complex number $x+iy$ for addition and multiplication. The author extends this to nth order matrices, showing that if $S(\theta)=e^{i\theta}A+e^{-i\theta}(I-A)$, A being any idempotent matrix, I the unit matrix, then $S(\alpha)S(\beta)=S(\alpha+\beta)$. He then uses such matrices to write a matrix, each of whose elements can be expanded in a Fourier series, in the form $\sum_k B_k S(k\theta)$, the B_k being the matrices of the Fourier coefficients. *L. C. Hutchinson* (Brooklyn, N. Y.).

✱**Mattioli, Ennio. Sull'algebra delle matrici permutabili con una matrice assegnata.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 139–146. Edizioni Cremonense, Rome, 1942.

The results obtained are special cases of the general theorem, apparently unknown to the author, that any matrix which is commutative with every matrix commutative with A is a scalar polynomial in A. [Cf. Wedderburn, Lectures on Matrices, Amer. Math. Soc. Colloquium Publ., v. 17, New York, 1934, p. 106.] *N. H. McCoy.*

Vasileios, Filon. Remarks on some criteria for the irreducibility of equations. Bull. Soc. Math. Grèce **22**, 181–190 (1946). (Greek)

Abstract Algebra

✱**Brauer, Richard. On some developments of modern algebra.** Proc. First Canadian Math. Congress, Montreal, 1945, pp. 183–205. University of Toronto Press, Toronto, 1946. $3.25.

✱**Birkhoff, Garrett. Universal algebra.** Proc. First Canadian Math. Congress, Montreal, 1945, pp. 310–326. University of Toronto Press, Toronto, 1946. $3.25.

A summary of known results on "universal algebra." The author also mentions several unsolved problems.

N. H. McCoy (Northampton, Mass.).

Grayev, M. Direct sums of cycles in the Dedekind structures. Rec. Math. [Mat. Sbornik] N.S. **19(61)**, 439–450 (1946). (Russian. English summary)

The paper deals with complete Dedekind structures in A. Kurosch's sense [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **7**, 185–202 (1943); these Rev. **6**, 145]. Properties of cycles of finite or transfinite length are considered. It is proved that, if the unit is a direct sum of cycles, then any cycle whose length is equal to the maximum of the lengths of the components is a direct summand of the unit. Further results are obtained by assuming: (1) every element is a sum of cycles; (2) if a/b contains a single point (i.e., a cycle of length 1 over b), then a/b is a cycle; (3) if a point of the structure is contained in the sum of a set S of cycles, then a finite subset of cycles can be chosen in S whose sum contains the given point. It is proved that, if the unit is a direct sum of a finite number of cycles of length τ, every cycle of length $\alpha<\tau$ is a subcycle of a cycle of length τ. Conditions are found for decomposability of elements into the direct sum of cycles; in this connection a theorem of R. Baer [Trans. Amer. Math. Soc. **52**, 283–343 (1942); these Rev. **4**, 109] is generalized. A well-known theorem concerning decomposition of certain infinite periodic Abelian groups into the direct sum of primary cyclic groups follows as a simple corollary. *J. Levitzki.*

Barbilian, D. Metrisch-konkave Verbände. Disquisit. Math. Phys. **5**, 1–63 (1946).

A "roughly concave" lattice is defined as one in which (i) with each $a<b$, a "length" $\lambda(a, b)>0$ is associated, (ii) if $a<b<c$, then $\lambda(a, b)+\lambda(b, c)\leqq\lambda(a, c)$. Distance $\rho(x, y)$ is defined by $\rho(x, x)=0$ and $\rho(x, y)=\lambda(x\cap y, x\cup y)$ if $x\neq y$; this defines a metric space. A detailed discussion is given of various types of "normal" elements, mostly following O. Ore [Trans. Amer. Math. Soc. **41**, 266–275 (1937), in particular, p. 271]. "Torsion numbers" $h_1(a, b)=\lambda(a, a\cup b)-\lambda(a\cap b, b)$ and $h_2(a, b)=h_1(b, a)$ are introduced. The paper leads to various partial generalizations of theorems of V. Glivenko [Amer. J. Math. **58**, 799–828 (1936); **59**, 941–956 (1937)] to roughly concave lattices. *G. Birkhoff.*

Schützenberger, Maurice-Paul. Sur certains treillis gauches. C. R. Acad. Sci. Paris **224**, 776–778 (1947).

Various facts are stated about "skew lattices," or systems with a binary, idempotent and associative multiplication, and elements O, I [cf. Klein-Barmen, Math. Z. **46**, 472–480 (1940); these Rev. **1**, 327]. The quasi-orderings defined by letting $a\leqq b$ mean $ab=a$, or $ab=b$, or both, are set up; so are the associated equivalence relations. Some consequences of assuming $aba=ab$ are mentioned; thus, in this case, any finitely generated skew lattice is finite. In the lattice of all identities on skew lattices, the "maximal" identity $ab=ba$ "covers" $aba=ab$ and $aba=ba$.

G. Birkhoff (Cambridge, Mass.).

Kolchin, E. R. Extensions of differential fields. III. Bull. Amer. Math. Soc. **53**, 397–401 (1947).

This paper is a continuation of part I [Ann. of Math. (2) **43**, 724–729 (1942); these Rev. **4**, 72], in which the notions of resolvent, order and dimension, as introduced by Ritt, are generalized for abstract differential fields. The author defines the notion of a resolvent of a prime differential ideal π in a ring $F\{y_1, \cdots, y_n\}$ (where F is an ordinary differential field and $y_1, \cdots, y_n$ unknowns); it is proved that, if F is extended to a larger differential field G, the ideal decomposition of the perfect ideal generated by π in $G\{y_1, \cdots, y_n\}$ runs parallel to the factorization of a resolvent. The paper also contains a correction to a proof contained in part II [Ann. of Math. (2) **45**, 358–361 (1944); these Rev. **5**, 227].

C. Chevalley (Princeton, N. J.).

Jacobson, N. On the theory of primitive rings. Ann. of Math. (2) **48**, 8–21 (1947).

La structure des anneaux simples sans condition minimale mais admettant des idéaux minimaux a été déterminée par le rapporteur (comme conséquence de résultats plus généraux) [Bull. Soc. Math. France **70**, 46–75 (1942); ces Rev. **6**, 144]; elle a été retrouvée indépendamment dans un cas particulier par B. H. Arnold [Ann. of Math. (2) **45**, 24–49 (1944); ces Rev. **5**, 147], et dans le cas général par T. Nakayama [Proc. Imp. Acad. Tokyo **20**, 61–66 (1944); ces Rev. **7**, 236] et enfin par N. Jacobson [Trans. Amer. Math. Soc. **57**, 228–245 (1945); Amer. J. Math. **67**, 300–320 (1945); ces Rev. **6**, 200; **7**, 2; nous citerons ces travaux J–I et J–II]. Le rapporteur a montré [loc. cit.] que tout anneau simple ayant des idéaux minimaux est isomorphe à un anneau $F(E, E')$ défini comme suit: E est un espace vectoriel (de dimension quelconque) sur un corps (en général non commutatif) K, E' un sous-espace du dual E^* de E (espace des formes linéaires sur E) tel que la relation $x'(x)=0$ pour tout $x'\epsilon E'$ entraîne $x=0$ dans E; si $\sigma(E, E')$ est la topologie sur E pour laquelle les voisinages de 0 sont les intersections finies d'hyperplans $x'(x)=0$ (avec $x'\epsilon E'$), $F(E, E')$ est l'anneau des applications linéaires u de E dans E, continues pour la topologie $\sigma(E, E')$ et telles que $u(E)$ soit de dimension finie. Dans J–I et J–II, l'auteur a abordé l'étude d'une classe d'anneaux plus généraux, les anneaux primitifs: un tel anneau peut être caractérisé comme sous-anneau partout dense d'un anneau $\mathfrak{L}(E)$ de tous les endomorphismes d'un espace vectoriel E, quand on munit E de la topologie discrète et $\mathfrak{L}(E)$ de la topologie de la convergence simple.

Dans le travail actuel, l'auteur montre qu'un anneau primitif A ayant des idéaux minimaux est isomorphe à un anneau d'endomorphismes $\bar{A}$ de E, continus pour une topologie $\sigma(E, E')$, $\bar{A}$ contenant $F(E, E')$, qui en est le socle (au sens du rapporteur) et est identique à l'intersection de $\bar{A}$ et de $F(E, E^*)$. Il obtient aussi diverses propriétés des idéaux à droite et des idéaux à gauche de $\bar{A}$, notamment en ce qui concerne les fermetures de ces idéaux lorsque $\bar{A}$ est muni de la topologie de la convergence simple dans E (ce dernier muni de la topologie discrète); ces propriétés généralisent des résultats analogues établis par le rapporteur [loc. cit.] lorsque A est un anneau simple ayant des idéaux minimaux. *J. Dieudonné* (São Paulo).

Goldman, Oscar. A characterization of semi-simple rings with the descending chain condition. Bull. Amer. Math. Soc. **52**, 1021–1027 (1946).

Eine homomorphe Abbildung ρ eines Schiefringes A in den Operatorenring einer Abelschen (additiven) Gruppe $\mathfrak{M}$ heisst Darstellung von A. Die Gruppe $\mathfrak{M}$ ist ein A-Modul gemäss der Festsetzung $am=\rho(a)m$ und umgekehrt definiert jeder A-Modul eine Darstellung von A. Alle Elemente aus A, für die $aM=0$, bilden ein zweiseitiges Ideal von A, den Annullator von $\mathfrak{M}$; $\mathfrak{M}$ heisst einfacher A-Modul, wenn 0 und $\mathfrak{M}$ die einzigen invarianten Teilmoduln sind. Als Radikal von A wird der Durchschnitt $\mathfrak{N}$ aller Annullatoren von einfachen A-Moduln bezeichnet. Satz 1: $\mathfrak{N}$ enthält jedes nilpotente Ideal. Es ist selbst nilpotent, wenn die Minimalbedingung für Linksideale erfüllt ist.

Der A-Modul $\mathfrak{M}$ heisst halbeinfach, wenn er die direkte Summe von (endlich oder unendlich vielen) einfachen Teilmoduln ist. Er heisst ein treuer A-Modul, wenn sein Annullator verschwindet, d.h. wenn es sich um eine isomorphe Abbildung von A auf gewisse lineare Transformationen von $\mathfrak{M}$ handelt. Bei Anwendung der Multiplication mit Elementen aus A von links her ist A selbst ein A-Modul; A heisst halbeinfach, wenn das Radikal verschwindet. Diese Definition steht im Einklang mit der vorhin gegebenen Definition des halbeinfachen A-Moduls. Der Faktorring von A nach $\mathfrak{N}$ ist stets halbeinfach. Satz 2: A ist genau dann halbeinfach, wenn A einen treuen halbeinfachen A-Modul besitzt [Verallgemeinerung eines Satzes von Weyl, The Classical Groups, Princeton University Press, 1939, S. 85; diese Rev. **1**, 42]. Als trivialer invarianter Teilmodul eines A-Modules $\mathfrak{M}$ wird der Modul aller m aus $\mathfrak{M}$, für die $Am=0$ ist, bezeichnet. Satz 3: A ist genau dann halbeinfach mit Minimalbedingung für Linksideale, wenn jeder A-Modul die direkte Summe des trivialen Teilmoduls und von endlich oder unendlich vielen einfachen A-Moduln ist. Diese Bedingung kann nach Satz 4 durch die Forderung, dass jedes Linksideal von A die Form eA mit $ee=e$ hat, ersetzt werden. *H. Zassenhaus* (Hamburg).

Goldman, Oscar. Semi-simple extensions of rings. Bull. Amer. Math. Soc. **52**, 1028–1032 (1946).

Im Anschluss an die vorherige Arbeit [siehe vorstehendes Referat] entsteht die Frage, wie ein Schiefring A beschaffen sein muss, der sich in einen halbeinfachen Schiefring B einbetten lässt, der also eine halbeinfache Erweiterung gestattet. Sei $E(\mathfrak{M})$ der Operatorenring des Moduls $\mathfrak{M}$; $\mathfrak{M}$ heisst quasieinfach, wenn $\mathfrak{M}$ ein einfacher $E(\mathfrak{M})$-Modul ist. Der Durchschnitt der Annullatoren aller A-Moduln $\mathfrak{M}$, die als Modul betrachtet quasieinfach sind, wird als das Erweiterungsradikal von A bezeichnet. Zu den A-Moduln, die als Modul betrachtet quasieinfach sind, gehören gemäss Hilfssatz 1 die einfachen A-Moduln. Nach Hilfssatz 2 sind die quasieinfachen Moduln Vektormoduln von endlicher oder unendlicher Dimension über einem Körper und umgekehrt Satz 1: A gestattet genau dann eine halbeinfache Erweiterung, wenn sein Erweiterungsradikal verschwindet. Bezeichnen wir mit T die in der additiven Gruppe von A enthaltene Untergruppe der Elemente von endlicher Ordnung (Torsionsmodul), so ist das Erweiterungsradikal von A gleich dem Durchschnitt von T mit allen Moduln pA (p durchläufe alle Primzahlen). Unter welcher Bedingung gestattet jeder Schiefring, der eine gegebene Abelsche Gruppe G als additive Gruppe besitzt, eine einfache Erweiterung? Nach Satz 4 genau dann, wenn für jede Primzahl p aus $p^2x=0$ stets $px=0$ folgt. *H. Zassenhaus.*

Brown, Bailey, and McCoy, Neal H. Radicals and subdirect sums. Amer. J. Math. **69**, 46–58 (1947).

Assume that there has been attached to every ring R a mapping $a\rightarrow F(a)$ of R into the set of its two-sided ideals in such a way that, if $a\rightarrow\bar{a}$ is a homomorphism of R onto another ring $\bar{R}$, then $F(\bar{a})=\overline{F(a)}$. To any such mapping F the authors attach an F-radical. An element $a\epsilon R$ is called F-regular if $a\epsilon F(a)$; a two-sided ideal is called F-regular if its elements are all F-regular. The F-radical of R is defined to be the set of elements which generate two-sided F-regular ideals. The vanishing of this F-radical for a ring R is necessary and sufficient for R to be isomorphic to a subdirect sum of subdirectly irreducible rings with zero F-radical. If we define $G(a)$ to be the two-sided ideal generated by the elements $ax-x$, $x\epsilon R$, then the vanishing of the G-radical, together with the ascending chain condition, implies the semi-simplicity of the ring. *C. Chevalley.*

Cohen, I. S., and Kaplansky, Irving. Rings with a finite number of primes. I. Trans. Amer. Math. Soc. **60**, 468–477 (1946).

Integral domains R (with unit elements) are considered which satisfy the following conditions: (a) every element is expressible in at least one way (but possibly in several ways) as a product of prime elements; (b) there are only a finite number of nonassociated prime elements. It is first proved that any such ring is a Noetherian ring in which there are only a finite number of prime ideals (they are all maximal). The study of the structure of R is then reduced to the case where R is a local ring. Let then M be the ideal of nonunits. The field R/M is proved to be finite; the number of its elements will be denoted by N. Let also k be the dimension of M/M^2, considered as a vector space over R/M; then it is shown that $(N^k-1)/(N-1)\leqq n$, where n is the number of distinct primes. The case where n is a prime number is discussed further. In this case, the multiplicative structure of R modulo units is uniquely determined; the ring R itself is uniquely determined if it is complete and of characteristic not 0. It is shown that, if N is a power of a prime and $n=(N^k-1)/(N-1)$ with $k\geqq 2$, then there exists a local ring R with n primes, for which R/M has N elements and M/M^2 is of dimension k over R/M, and in which every element of M^2 is divisible by every prime.

C. Chevalley (Princeton, N. J.).

Kaplansky, Irving. Topological rings. Amer. J. Math. **69**, 153–183 (1947).

In the first part of this paper structure theorems for compact and locally compact rings are obtained. Jacobson [same J. **67**, 300–320 (1945); these Rev. **7**, 2] introduced a new definition of radical which allows the development of a structure theory for arbitrary rings. In particular, in the case of a normed ring this radical is identical with the generalised radical of Gelfand [Rec. Math. [Mat. Sbornik] (N.S.) **9**(**51**), 3–24 (1941); these Rev. **3**, 51]. Jacobson's theory is here summarized and applied to topological rings. Several new concepts are introduced, e.g., Q_r-rings in which the right quasi-regular elements form an open set; Q_l-rings are defined similarly and Q-rings are both Q_r and Q_l-rings. However, the author has not so far been able to find a Q_r-ring which is not also Q_l. All locally compact rings without divisors of zero and all complete metric rings are Q-rings. Another definition used is that of abstract boundedness which goes back to Shafarevich [C. R. (Doklady) Acad. Sci. URSS (N.S.) **40**, 133–135 (1943); these Rev. **6**, 164] and which ensures that every compact set is bounded. An element x is called nilpotent if $x^n\to 0$, a ring is a nil-ring if it contains only nilpotent elements and it is called nilpotent if any neighbourhood of 0 contains a power of the radical R. For compact rings it is shown, among many other things, that the radical of a totally disconnected ring is nilpotent; furthermore, the radical is the union of all nil right and left ideals. A semi-simple compact ring is isomorphic and homeomorphic to a Cartesian direct sum of finite simple rings. For rings with radical a structure theorem is proved in the commutative case, namely that the ring is the Cartesian direct sum of a ring which coincides with its radical and rings with units which are simple modulo their radical. Various other results are obtained for compact and finally locally compact rings [cf. Jacobson and Taussky, Proc. Nat. Acad. Sci. U. S. A. **21**, 106–108 (1935)]. The theorem obtained there that a compact field is finite is now generalised and it is proved that a compact ring with no proper closed ideals is finite. The paper includes several remarks concerning the continuity of the inverse and quasi-inverse in a ring which in some cases is already implied by other continuity assumptions. In this connection generalisations of work by Otobe [Jap. J. Math. **19**, 189–202 (1945); these Rev. **7**, 237] and of D. Montgomery [Bull. Amer. Math. Soc. **42**, 879–882 (1936)] are obtained.

The second part of the paper deals with the ring $C(X, A)$ of continuous functions from a topological space X to a topological ring A. The case when A is the field of real or complex numbers was studied by Gelfand and Kolmogoroff [C. R. (Doklady) Acad. Sci. URSS (N.S.) **22**, 11–15 (1939)], the case where X is totally disconnected and A the field of integers mod 2 by Stone [Trans. Amer. Math. Soc. **41**, 375–481 (1937)], whose work was generalised by Jacobson [Proc. Nat. Acad. Sci. U. S. A. **31**, 333–338 (1945); these Rev. **7**, 110]. In particular, it is shown that a totally disconnected X can be completely characterised by the topological and algebraic properties of $C(X, A)$ if A is simple and the topology of $C(X, A)$ is based on uniform convergence on compact subsets of X. The algebraic properties of $C(X, A)$ only suffice if X is also compact. One of the main problems is the investigation of the maximal ideals. Conditions on X and A sufficient to ensure that the prime ideals in $C(X, A)$ are maximal are given. An example is shown in which not all prime ideals are maximal and the importance of these conditions is thus emphasized. The largest two-sided ideals contained in a maximal ideal of a ring with unit have been studied by Jacobson and called primitive. The space formed by these ideals with Stone's topology (the structure space) is determined for the case that X is totally disconnected and compact. For such X and if A is a Q-ring it is shown that if all prime ideals in A are maximal the same is implied for $C(X, A)$. A question raised by Jacobson as to whether the prime ideals of a certain subring of $C(X, A)$ (A is assumed a discrete division ring) are maximal is answered affirmatively by showing that this subring coincides in fact with $C(X, A)$. A special study is made of functions with values in certain division rings, of functions vanishing at a point and of bounded functions.

O. Todd-Taussky (London).

Barsotti, I. Studi sopra le algebra senza base finita. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 1187–1189 (1946).

A brief account of results which are to appear elsewhere.

Dubisch, Roy. The Wedderburn structure theorems. Amer. Math. Monthly **54**, 253–259 (1947).

Expository article.

★Svartholm, Nils. On Clifford's algebra over the field of the real numbers. C. R. Dixième Congrès Math. Scandinaves 1946, pp. 281–289. Jul. Gjellerups Forlag, Copenhagen, 1947.

If Q is a nondegenerate quadratic form in n variables over a field F of characteristic different from 2, then Q determines a Clifford algebra over F. The author studies the case that F is the field of real numbers. It is shown that, if s is the signature of Q, then in the cases $n\equiv 0 \pmod 2$, $s\equiv 0$ or $2 \pmod 8$, the algebra is a complete matric algebra over the field of real numbers; in the cases $n\equiv 0 \pmod 2$, $s\equiv 4$ or $6 \pmod 8$, the algebra is a complete matric algebra over the quaternions; in the case $n\equiv 1 \pmod 2$, $s\equiv 3 \pmod 4$ the algebra is a complete matric algebra over the field of

complex numbers. In the remaining case $n\equiv 1 \pmod 2$, $s\equiv 1 \pmod 4$, the algebra is semisimple but not simple.

R. Brauer (Toronto, Ont.).

Levitzki, J. On a problem of A. Kurosch. Bull. Amer. Math. Soc. 52, 1033–1035 (1946).

Le problème de Kurosch est le suivant: si une algèbre A sur un corps K est telle que chacun de ses éléments satisfasse à une équation algébrique de degré borné, à coefficients dans K, et si en outre A est engendré par un nombre fini d'éléments, A est-elle de rang fini sur K? N. Jacobson [Ann. of Math. (2) **46**, 695–707 (1945); ces Rev. **7**, 238] a ramené le problème au cas des nil-algèbres (algèbres dont tout élément est nilpotent), et I. Kaplansky [même Bull. **52**, 496–500 (1946); ces Rev. **8**, 63] a donné une solution partielle du problème dans ce cas. L'auteur résoud complètement le problème de Jacobson (et par suite celui de Kurosch) en démontrant plus généralement que tout anneau A, tel qu'il existe un entier n pour lequel $a^n=0$ pour tout $a\epsilon A$, est semi-nilpotent, c'est-à-dire [Levitzki, même Bull. **49**, 461–466 (1943); ces Rev. **4**, 238] que tout sous-anneau de A engendré par un nombre fini d'éléments est nilpotent. La démonstration procède comme suit: si $n(A)$ est la plus petite valeur de n telle que $a^n=0$ pour tout $a\epsilon A$, l'auteur prouve que si A n'était pas semi-nilpotent, on pourrait définir un anneau B qui n'est pas semi-nilpotent, et pour lequel $n(B)<n(A)$.

Dans le compte-rendu du travail précité de I. Kaplansky, le rapporteur avait signalé que, d'après le compte-rendu d'un article de A. Malcev [Rec. Math. [Mat. Sbornik], N.S. **13**(55), 263–286 (1943); ces Rev. **6**, 116] ce dernier aurait aussi résolu le problème de Kurosch. En réalité, comme N. Jacobson l'a signalé [dans une lettre au rapporteur] le résultat obtenu par Malcev est moins général que le théorème de Jacobson-Kaplansky-Levitzki, car il ne s'applique qu'aux algèbres à un nombre fini de générateurs représentables comme sous-algèbres d'une algèbre de matrices sur un corps commutatif; or, contrairement à ce qui est indiqué par erreur dans le compte-rendu de l'article de Malcev, il y a des algèbres à un nombre fini de générateurs qui ne sont pas représentables de cette manière.

J. Dieudonné (São Paulo).

Albert, A. A. The Wedderburn principal theorem for Jordan algebras. Ann. of Math. (2) 48, 1–7 (1947).

In continuation of his earlier work on Jordan algebras [Trans. Amer. Math. Soc. **59**, 524–555 (1946); these Rev. **8**, 63] the author proves the following analogue of a theorem of Wedderburn on associative algebras. If $\mathfrak{A}$ is a Jordan algebra over a nonmodular field and if $\mathfrak{A}$ is neither semisimple nor solvable, then A can be represented as a sum $\mathfrak{S}+\mathfrak{N}$ of the radical $\mathfrak{N}$ of $\mathfrak{A}$ and a semi-simple algebra S which then is isomorphic to $\mathfrak{A}-\mathfrak{N}$.

R. Brauer.

Guarnaccia, Clelia. Sulle algebre complesse commutative irriducibili del 4.° ordine dotate di modulo. Rend. Accad. Sci. Fis. Mat. Napoli (4) 9, 45–54 (1939).

For each of the four irreducible commutative algebras of order 4 with unit over the complex numbers, the author proves that a "general" equation of degree n has n roots. [A more general result of this kind was given later by Spampinato, Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 96–104; these Rev. **8**, 251]. The paper also contains a discussion of the region of convergence of a power series in each of the four algebras.

I. Kaplansky (Chicago, Ill.).

Chevalley, Claude. Algebraic Lie algebras. Ann. of Math. (2) 48, 91–100 (1947).

For any $n\times n$ matrix X and polynomial $p(y_{11}, \cdots, y_{nn})$, let $p\rho X$ mean that $p(y_{11}, \cdots, y_{nn})=0$ for any power $Y=e^{tX}$ of X. Given a Lie algebra L of $n\times n$ matrices with coefficients in a field K of characteristic zero (infinity), we may define $L^\dagger$ as the set (ideal) of all polynomials p such that $p\rho X$ for all $X\epsilon L$; dually, we may define, for any polynomial ideal J, the set (Lie algebra) of all $n\times n$ matrices X such that $p\rho X$ for all $p\epsilon J$. Then, by definition, $L=(L^\dagger)^*$ if and only if L is an "algebraic Lie algebra." The author proves that, for any L, every ideal (invariant subalgebra) of L is one of $(L^\dagger)^*$; the center of $(L^\dagger)^*$ contains the center of L; L and $(L^\dagger)^*$ have the same derived algebra; etc. The set of derivations of any linear algebra (associative or not) is an algebraic Lie algebra. If, for any $X\epsilon L$, every XY has a zero trace (Y any $n\times n$ matrix), then L is solvable.

The preceding results are applied to derive the following facts of representation theory. The adjoint representation of any semi-simple Lie algebra L is semi-simple; the adjoint algebra of L is the set of its derivations (i.e., every infinitesimal automorphism is inner). Every irreducible representation of a solvable Lie algebra is Abelian. The radical H of any algebraic Lie algebra is algebraic and can be represented as the direct sum of the nilpotent matrices of H and an Abelian algebra of "semi-simple" matrices. (This proof uses the theorem of Levi and J. H. C. Whitehead.) The proofs are characterized by reliance on purely "rational" methods.

G. Birkhoff (Cambridge, Mass.).

THEORY OF GROUPS

***Miller, George Abram. Collected Works, vol. 3.** University of Illinois Press, Urbana, 1946. xi+499 pp. $7.50.

Starting with an article [no. 172] on the history of group theory from 1908 to 1915, this third volume of the author's collected works includes 89 papers and lists 91 papers which were not reprinted. At least fifteen of the printed papers are concerned with a study of groups defined by two or more operators satisfying stated conditions. Others are concerned with groups of isomorphisms, with holomorphs or with representation groups of a given group. The cosets of a group, substitution groups of small degree, groups having a small number of sets of conjugate operators, and multiply transitive groups each have articles devoted to them. A list is given [no. 189] of all groups of order less than or equal to 12 which can be formed by prime residues with respect to modular systems. In 1909 [no. 190] the author solved the problems of listing all groups whose orders are the eleven numbers 1909 to 1919 inclusive, concluding with the remark that the next number 1920 is the order of more groups than all the ten orders under consideration. Papers of interest to the general reader include, besides the history mentioned above, articles on "The founder of group theory" [no. 207], "Some thoughts on modern mathematical research" [no. 221] and "Remarks on the bearing of the theory of groups" [no. 248]. There are many references in footnotes to the periodical literature about group theory. [The second volume of these collected works was published in 1939 and reviewed in these Rev. 1, 43.]

J. S. Frame (East Lansing, Mich.).

Koliankowsky, D. Sur les sous-groupes non-spéciaux des groupes finis. Rec. Math. [Mat. Sbornik] N.S. 19(61), 429–437 (1946). (Russian. French summary)

The following theorem is proved. If the order of a finite nonsolvable group is divisible by n different primes, then it contains at least n nonspecial subgroups which are mutually nonisomorphic. (A group is called special if it is the direct product of its Sylow subgroups.) An immediate consequence of this theorem is the following result due to O. Schmidt [same Rec. 31, 366–372 (1924)]: a finite group is solvable if all its proper subgroups are special.

J. Levitzki (Jerusalem).

Fu, C. S. On Frobenius' theorem. Quart. J. Math., Oxford Ser. 17, 253–256 (1946).

Als Verallgemeinerung eines Satzes von Frobenius wird bewiesen, dass die Anzahl der Lösungen der Gleichung $x^b=1$ in einer Gruppe der Ordnung ab genau gleich b ist, wenn b teilerfremd zu $a\phi(a)$ ist und wenn keine der Sylowgruppen, deren Ordnung in a aufgeht, zwei verschiedene, aber in G konjugierte zyklische Untergruppen enthält.

H. Zassenhaus (Hamburg).

Kaloujnine, Léo. Sur le groupe $\mathcal{P}_\infty$ des tableaux infinis. C. R. Acad. Sci. Paris 224, 1097–1099 (1947).

Various properties of the Sylow subgroups P_m of the symmetric group of degree p^m have been discussed by the author in previous notes [cf. the same C.R. 224, 253–255 (1947); these Rev. 8, 367, for an explanation of the notation used here]. In the present note the author shows that suitable homomorphisms exist among the groups P_m for the formation of the "projective limit" P_∞ of these groups. The elements of the group P_∞ may be represented by symbols $[a, a(X_1), a(X_2), \cdots]$ having an infinity of terms but with a law of multiplication similar to that of the finite symbols $[a, a(X_1), \cdots, a(X_{m-1})]$ used previously. Several properties of the group P_∞ are mentioned, among them the following. It is a complete compact topological group if a fundamental system of neighbourhoods of the identity is taken as the sets of elements of depth not less than δ, $\delta=1, 2, 3, \cdots$. If we define a group G to be a "p_∞-group" when it contains a decreasing series $G=H_1\supset H_2\supset\cdots$ of normal subgroups whose indices in G are powers of p and whose intersection consists only of the unit element, and if $\bar{G}$ is the completion of such a p_∞-group G in the topology defined by taking the subgroups H_k as a fundamental system of neighbourhoods, then P_∞, itself a complete p_∞-group, is a universal group for p_∞-groups in the sense that every $\bar{G}$ is isomorphic to some closed subgroup of P_∞. The group P_∞ itself is isomorphic to the maximal p_∞-subgroups of the group T of all isometric mappings of the p-adic integers into themselves, all such maximal p_∞-subgroups being conjugate in T.

S. A. Jennings (Vancouver, B. C.).

Kulakoff, A. A. Sur les groupes d'ordre impair. C. R. (Doklady) Acad. Sci. URSS (N.S.) 53, 683–685 (1946).

After studying the quasi-transform RHR of a subgroup H by an element R not in H and defining a quasi-normal divisor of G to be a subgroup H of G such that $RHR=H$ for each R in G, the author shows that a group of odd order has no proper quasi-normal subgroups. If P is an element of least prime order p in a group G of odd order, the normalizer of the subgroup $\{P\}$ is the same as the normalizer N_P of the element P; the elements of G not in N_P fall into double cosets with respect to $\{P\}$, not self inverse, and each containing p^2 elements.

J. S. Frame (East Lansing, Mich.).

Kulakoff, A. A. Sur la représentation régulière d'un p-groupe. C. R. (Doklady) Acad. Sci. URSS (N.S.) 54, 113–116 (1946).

In five previous papers [Rec. Math. [Mat. Sbornik] N.S. 2(44), 357–359, 1003–1006 (1937); 3(45), 187–189 (1938); 4(46), 371–373 (1938); 8(50), 69–72 (1940); these Rev. 2, 126], the author investigated the regular representation of a finite abstract group. In the present paper the study of a special group of order $81=3^4$ leads him to a more detailed consideration of the structure of the permutations of the (right) regular representation Γ of an arbitrary p-group G of order p^m (p prime). Denote by (R) the permutation in Γ which represents the element R of G. Then the principal result can be stated as follows. The elements of G considered as objects permuted by permutations in Γ can be simply taken as the numerals $0, 1, \cdots, p^m-1$ in a certain order such that for an arbitrary (R) of Γ the following two conditions are satisfied. (1) If $(a_1, \cdots, a_n)$ is any cycle of (R), then $a_1, \cdots, a_n, a_1$ mod p form an arithmetical progression mod p with a common difference which is the same for all the cycles of (R). (2) Either all the integers $[a_i/p]$ for $i=1, \cdots, n$ belongs to the same residue class mod p or they are equally distributed into all the p distinct residue classes; the alternative to be taken is the same for all the cycles of (R). The proof is based on induction on n by making use of the factor group $G/\{P\}$ of G modulo the cyclic subgroup $\{P\}$ generated by an element P of order p in the center of G.

H. F. Tuan (Peiping).

Dietzmann, A. P. On an extension of Sylow's theorem. Ann. of Math. (2) 48, 137–146 (1947).

Let π be a given finite or infinite set of primes and let π' be the set of integers each of which is the product of a finite number of powers of primes belonging to π; let H denote a given set (not necessarily finite or denumerable) of subgroups $\mathfrak{H}_\alpha$ of a group $\mathfrak{G}$. An element $a\epsilon\mathfrak{G}$ will be called a $|\pi, H|$-element of $\mathfrak{G}$ if for every $\mathfrak{H}_\alpha$ of the set H there exists an integer δ_α of the set π' such that $a^{\delta_\alpha}\epsilon\mathfrak{H}_\alpha$; a subgroup $\mathfrak{P}$ of $\mathfrak{G}$ is a $|\pi, H|$-subgroup of $\mathfrak{G}$ if every element of $\mathfrak{P}$ is a $|\pi, H|$-element of $\mathfrak{G}$. In this notation, therefore, the Sylow subgroups of a group are maximal $|\pi, H|$-subgroups, where π consists of the single prime p and H consists of one subgroup, $\{1\}$, i.e., $|p, 1|$-subgroups. In the first part of the paper the author studies $|\pi, H|$-subgroups for an arbitrary set π and, in particular, obtains sufficient conditions that such a subgroup should also be a $|\pi, N|$- or a $|\pi, \mathfrak{D}|$-subgroup, where N is the set of normalisors and $\mathfrak{D}$ the intersection of the subgroups of H. The most important results are obtained, however, for the case when π consists of the single prime p. The nature of these results is indicated by the following two theorems, both of which may be considered as generalizations of an earlier result on Sylow subgroups obtained by the author and others [C. R. (Doklady) Acad. Sci. URSS (N.S.) 15, 71–76 (1937); Rec. Math. [Mat. Sbornik] N.S. 3(45), 179–184 (1938); cf. R. Baer, Duke Math. J. 6, 598–614 (1940); these Rev. 2, 2]. If the subgroups $H=\{\mathfrak{H}_\alpha\}$ of $\mathfrak{G}$ are conjugate, and if $\mathfrak{G}$ contains a finite set $\mathfrak{P}_1, \cdots, \mathfrak{P}_k$ of conjugate maximal $|p, H|$-subgroups such that every subgroup of the form $\{a_i, h_\alpha\}$ is a $|p, H|$-subgroup, where $a_i\epsilon\mathfrak{P}_i$ and $h_\alpha\epsilon\mathfrak{H}_\alpha$, then every maximal $|p, H|$-subgroup of $\mathfrak{G}$ is one of the $\mathfrak{P}_i$ and $k\equiv 1 \pmod p$. If $H=\{\mathfrak{H}_1, \cdots, \mathfrak{H}_k\}$ is a set of normal subgroups of $\mathfrak{G}$, and if $\{\mathfrak{P}_\beta\}$ is a (not necessarily finite) set of conjugate maximal $|p, H|$-subgroups of $\mathfrak{G}$ such that the set of subgroups of the form $\mathfrak{H}_i\mathfrak{P}_\beta$, $i=1, \cdots, k$, all β, is finite, then all maximal

$|p, \mathfrak{D}|$-subgroups of $\mathfrak{G}$, where $\mathfrak{D}$ is the intersection of the $\mathfrak{H}_i$, are to be found among the subgroups $\{\mathfrak{P}_\beta\}$.

S. A. Jennings (Vancouver, B. C.).

Ado, I. D. On locally finite p-groups with the minimality condition for normal divisors. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 471–473 (1946).

The following theorem is obtained: in a denumerably infinite, locally finite p-group the minimal condition for normal subgroups implies the minimal condition for all subgroups. The by no means elementary proof depends upon the following three lemmas. (i) If a locally finite p-group has an upper central series $Z_1 \subset Z_2 \subset \cdots$, then Z_2/Z_1 has no elements of infinite height. (ii) If a locally finite p-group is the extension of a complete Abelian p-group by a finite p-group, then the minimal condition for subgroups of this group is a consequence of the condition for normal subgroups. (iii) If a denumerably infinite p-group with an upper central series has no subgroup of finite index, it is Abelian. From these lemmas and an earlier result of the author [same C. R. (N.S.) **40**, 299–301 (1943); these Rev. **6**, 146] to the effect that a locally finite p-group with minimal condition for normal subgroups has a nontrivial centre, the proof of the theorem follows by induction. *S. A. Jennings.*

Artin, Emil. The free product of groups. Amer. J. Math. **69**, 1–4 (1947).

In this paper the author uses a new method to introduce the free product of groups. Let Σ be a set of groups Γ which can be assumed to be disjoint and whose elements will be denoted by small letters $a, b, \cdots$, whereas the capitals $A, B, \cdots$ will stand for formal products $A = a_1a_2 \cdots a_n$, to be said to be of length n. Two such products will be considered as equal when and only when they consist of the same factors in precisely the same order. (In other words, no simplification whatsoever will be allowed in the expression of a formal product.) Multiplication of two formal products is defined by writing first A and then B. In contrast with the traditional treatment, the associative law is obvious here while the question of equality requires careful consideration. For convenience, we introduce the empty product E and also a pure notation $A^{-1} = a_n^{-1} \cdots a_2^{-1}a_1^{-1}$; furthermore, if all a_i belong to the same group Γ and $a_1a_2 \cdots a_n$ computed in Γ has the value 1, then A is called an elementary expression. A set Δ of formal products is defined by induction such that (1) $E\epsilon\Delta$; (2) $n>0$, $A\epsilon\Delta$ if and only if $A = A_0a_1', A_1a_2', \cdots, A_{m-1}a_m'A_m$, where $A_i\epsilon\Delta$ and $a_1'a_2' \cdots a_m'$ is an elementary expression. We can show essentially by mathematical induction that (1) $A\epsilon\Delta$ implies $A^{-1}\epsilon\Delta$, (2) $A\epsilon\Delta$ and $B\epsilon\Delta$ imply $AB\epsilon\Delta$, (3) $A\epsilon\Delta$ implies $BAB^{-1}\epsilon\Delta$ and conversely, (4) $AB\epsilon\Delta$ implies $BA\epsilon\Delta$. Now if we define $A \equiv B$ as meaning $AB^{-1}\epsilon\Delta$, then we have an equivalence relation. Hence our formal products form a group if we use congruence instead of equality, and here the elements of Δ are precisely those congruent to E and $AA^{-1} \equiv E$. This group contains each Γ as a subgroup, with only their unit elements identified as 1. We can now define a normal expression as one where no factor is 1 and no two adjoining terms are of the same group. Then every A is congruent to a normal expression and two normal expressions are congruent if and only if they are identical. Thus in the sense of congruence we can always reduce a formal product to a normal expression and then deal with this simplified expression. *H. F. Tuan* (Peiping).

Baer, Reinhold. The double chain condition in cyclic operator groups. Amer. J. Math. **69**, 37–45 (1947).

An Abelian group A with a system M of operators (in short: M-group) is called cyclic if it is generated by a single element. An M-group is of finite rank r if there exists an integer r with the following property: if U is a finite subset of A, then there exists an M-subgroup of A which is generated by not more than r elements and contains U. In the special case where A is a ring with identity, A is a cyclic M-group with $M = A$ and one has Hopkins' theorem [Ann. of Math. (2) **40**, 712–730 (1939); these Rev. **1**, 2] which states that the double chain condition for admissible subgroups (in this case: right ideals) is a consequence of the descending chain condition. The author constructs an example which shows that Hopkins' theorem does not hold for general cyclic M-groups and establishes the following generalizations of this theorem. Let M be a ring with unity and $A = gM$ a cyclic M-group. If Z is a subring of M, denote by $Q(g, Z)$ the set of all elements z in Z satisfying $gz = 0$. Then the double chain condition is satisfied by the M-subgroups of A if there exists a subring Z of M which contains the unity of M and has the following properties: (1) $MQ(g, Z) \subseteq Q(g, Z)M$; (2) the descending chain condition is satisfied by the Z-subgroups of A. The assertion of the theorem remains true if condition (2) is replaced by the following two conditions: (2′) the descending chain condition is satisfied by the Z-subgroups of gZ; (3) M is of finite rank over Z (i.e., the addition group M_+ of M is a Z-group of finite rank). *J. Levitzki* (Jerusalem).

Kontorovitch, P. Sur les groupes à base de partition. II. Rec. Math. [Mat. Sbornik] N.S. **19**(61), 287–308 (1946). (Russian. French summary)

[Part I appeared in the same Rec. N.S. **12**(54), 56–70 (1943); these Rev. **5**, 144]. The present part is mainly devoted to the study of "isolated" subgroups of a given group. The subgroup H is isolated in G if H either contains or is mutually disjoint with any cyclic subgroup of G. If in this definition "cyclic" is replaced by "commutative," one obtains the definition of a "strongly isolated" subgroup. A group is "dense" if it has no proper isolated subgroups. These concepts play a fundamental part in the study of various decompositions of a group into the set-theoretical sum of proper subgroups, the components of the decomposition. If the components are mutually disjoint, the decomposition is called a "partition." Each component in a partition is an isolated subgroup; if the components of any decomposition are isolated and dense, then this decomposition is a partition. Conclusive results are obtained for Abelian groups, whose partitions were previously studied by J. W. Young [Bull. Amer. Math. Soc. **33**, 453–461 (1927)]. Other results are typified by the following examples. The central of a nondense periodic group is a p-group. An infinite group has at most one isolated dense subgroup of finite index. If in a periodic group each proper subgroup differs from its normalizer, then each isolated subgroup without partition is either cyclic of prime order or an invariant subgroup. If a group G can be decomposed into a finite number of components whose cross-cut is of finite order, then G itself is of finite order. The order of a strongly isolated subgroup of a finite group is prime to its index. A group is of the Frobenius type if and only if it contains a finite strongly isolated invariant subgroup. Finally, partitions modulo an invariant subgroup are studied.

J. Levitzki (Jerusalem).

Gelfand, I. M., and Neumark, M. A. On unitary representations of the complex unimodular group. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 195–198 (1946).

The authors announce a solution of the problem of determining all irreducible unitary representations of the complex unimodular group G, these representations being realized as operators in the space of functions of cosets with respect to certain subgroups of G. The present note describes (without proofs) certain coset spaces and the invariant measures on them. The subgroups of G which are considered are (1) unimodular matrices with zero everywhere below the main diagonal (upper triangular); (2) upper triangular matrices with main diagonal elements all unity; (3) lower triangular unimodular matrices; (4) lower triangular matrices with main diagonal elements all unity; (5) diagonal unimodular matrices. Both one-sided and two-sided cosets of these subgroups are considered. Proofs and more complete statement of the results are promised for later papers.

R. M. Thrall (Ann Arbor, Mich.).

Braconnier, Jean. Sur la notion de limite projective de groupes topologiques. C. R. Acad. Sci. Paris **224**, 370–372 (1947).

A generalization is given of the notion of inverse system of topological groups, and a corresponding limit group is defined. The main point of the generalization is that a projection $f_{\beta\alpha}$ of the system $(\alpha<\beta)$ does not map the entire group G_α into G_β but only a subgroup $H_{\alpha\beta}$ of G_α. The limit group is obtained as a factor group of a subgroup of the direct product $\prod G_\alpha$. If each G_α is locally compact Abelian, each $f_{\beta\alpha}$ is onto and has a compact kernel $K_{\alpha\beta}$, then the set of character groups $\hat{G}_\alpha$ and maps $K_{\alpha\beta}\rightarrow\hat{G}_\beta$ contragredient to the $f_{\beta\alpha}$ form a system of the same type and the two limit groups are character groups of one another. In this way both the group and its character group are obtained as the same type of limit of two "dual" systems. This is in contrast with the usual procedure of using inverse systems for compact groups and direct systems for discrete groups.

N. E. Steenrod (Princeton, N. J.).

Segal, I. E. The group algebra of a locally compact group. Trans. Amer. Math. Soc. **61**, 69–105 (1947).

This paper depends chiefly upon the following basic ideas. (a) On a locally compact group G, addition and convolution define the space L^1 (with respect to the Haar measure) as a Banach algebra A, called "the" group-algebra by the author. [The same holds for the algebra R of all Radon operators on G, of which A is a closed subalgebra; in some respects R is more convenient than A; e.g., it has a unit-element; an investigation of the structure of R, similar to that carried out by the author for A, might be of interest.] (b) Let G admit a (continuous) representation by bounded operators on a Banach space B; then, e.g., by means of the Bochner integral (of functions with values in B), this can be extended to a representation of A (and also to one of R). (c) In particular, in very general cases when B is a function-space over G, invariant by translations on G, this extension is analytically defined by the convolution (of functions in A with functions in B); this was well known in the case $B=L^p$.

Taking first $B=L^2$, the main idea in the classical proof of the Peter-Weyl theorem leads to the consideration of a bounded self-adjoint operator in L^2; when G is not compact, this operator is not (in general) completely continuous, which had hitherto stopped progress along those lines; however, from the existence of a nonempty spectrum for such an operator, the author succeeds in deducing that the algebra A is (in a certain "weak" sense) "semi-simple." When G is the product of an Abelian and a compact group, this is improved by showing that A is "strongly semi-simple," a fact closely related to the existence of "sufficiently many" representations of G by unitary matrices; in view of the above remarks, this, however, can hardly be considered as an independent proof for the existence of these representations. In the general case, one can only show that there exist "sufficiently many" representations by bounded operators in Banach spaces.

The "spectrum" is then defined as the set of all "regular" maximal ideals in A, with a suitable T_1-topology; this is shown to be discrete for a compact G, and homeomorphic to the dual of G when G is Abelian [it is doubtful whether any other case could be treated by similar methods]. In those cases, the author obtains a rather general theorem ("spectral resolution") for closed ideals of a certain type in A, with applications to Tauberian problems. Apart from a complicated theorem, of which the author states that it includes Wiener's theorem, and Pitt's extensions of it, as "very special cases," attention may be drawn to the following result: on the group of real numbers, if f is both in L^1 and in L^p $(1<p<+\infty)$, and such that the set of zeros of its Fourier transform is discrete, the translations of f span L^p. Part of the novelty of the author's treatment lies in his use of the Bochner integral, although verification of the various steps involved is mostly left to the reader and seems to be no easy task.

A. Weil (Chicago, Ill.).

Ambrose, W. Measures on locally compact topological groups. Trans. Amer. Math. Soc. **61**, 106–121 (1947).

Let G be a locally compact topological group with Haar measure m. A measure n is called a refinement of m $(n\geqq m)$ if every m-measurable set of positive m-measure contains an n-measurable set of positive n-measure. It is a strong refinement $(n\gg m)$ if every n-measurable set is of m-measure 0. Given a refinement n of m which is a Weil measure [see A. Weil, L'Intégration dans les groupes topologiques et ses applications, Actual. Sci. Ind., no. 869, Hermann, Paris, 1940, pp. 141–143; these Rev. 3, 198], the author constructs a new locally compact topology U_1 in G which is finer than the original topology of G and less fine than the Weil topology induced on G by n, and, in fact, is the finest locally compact topology with these two properties. The author then shows that (1) the U_1 topology is discrete if and only if every subset of an n-measurable set is n-measurable; (2) the U_1 locally compact subgroup generated by a U_1 compact neighborhood of the identity has m-measure 0 if and only if $n\gg m$. Thus a strong refinement of m leads to a nontrivial subgroup of G.

L. H. Loomis.

Ambrose, W. Direct sum theorem for Haar measures. Trans. Amer. Math. Soc. **61**, 122–127 (1947).

To prove (2) in the preceding review the author needs, and here proves, the following theorem. Let G be a locally compact topological group, G' a closed invariant subgroup of G and G_1 the corresponding quotient group. Let the left invariant Haar measures of G, G' and G_1 be denoted, respectively, by m, m' and m_1 and also use the notation m' for the transplanted measure on the cosets of G' in G. Then if E is any m-measurable set, the set $E\cap xG'$ is m'-measurable for m_1 almost all x, the function $m'(E\cap xG')$ is m_1-integrable and $mE=\int m'(E\cap xG')dm_1$. A different case of this theorem was previously proved by Weil [reference in the preceding review, pp. 42–45]. [The present proof as it stands is

incorrect, but becomes correct if the measure is restricted to the Borel field generated by those compact sets which can be expressed as a countable product of open sets.]

L. H. Loomis (Cambridge, Mass.).

Knichal, Vladimír. Sur une généralisation d'un théorème des MM. Blichfeldt et Visser dans la géométrie des nombres. Časopis Pěst. Mat. Fys. 71, 33–44 (1946). (French. Czech summary)

Knichal, Vladimír. Sur la distribution des mesures sur une sphère à n dimensions. Časopis Pěst. Mat. Fys. 71, 45–54 (1946). (French. Czech summary)

Let $H=G/g$ be a homogeneous space, defined by a compact group G and a subgroup g of G; let μ be a Radon measure on H and m the (uniquely determined) invariant measure on H (with respect to the transformations of G) such that $m(H)=1$. Then, if A is a Borel set in H,

$$\inf_{x\epsilon G} \mu(xA) \leqq \mu(H)\cdot m(A) \leqq \sup_{x\epsilon G} \mu(xA).$$

In fact, applying the Lebesgue-Fubini theorem to the product of the μ-measure on H and of the Haar measure on G, and to the function $\varphi_A(x^{-1}P)$ (where $x\epsilon G$, $P\epsilon H$ and $\varphi_A(P)$ is the characteristic function of A), one gets

$$\mu(H)\cdot m(A) = \int \mu(x^{-1}A)dx.$$

This and other very similar results are proved by the author, seemingly without any knowledge of the modern theory of topological groups, the main idea in his proof being essentially as indicated above. It is shown that various known results are either contained in the above or easily derived from it. *A. Weil* (Chicago, Ill.).

Malcev, A. On solvable topological groups. Rec. Math. [Mat. Sbornik] N.S. 19(61), 165–174 (1946). (Russian. English summary)

Généralisant un résultat annoncé par C. Chevalley [Lectures in Topology, pp. 291–292, University of Michigan Press, 1941; ces Rev. 3, 135], l'auteur décrit les groupes localement compacts, connexes et solubles comme des limites de groupes de Lie ou localement comme des produits directs de groupes compacts et de groupes de Lie.

H. Freudenthal (Amsterdam).

***Chevalley, Claude. La théorie des groupes de Lie.** Proc. First Canadian Math. Congress, Montreal, 1945, pp. 338–354. University of Toronto Press, Toronto, 1946. $3.25. Expository lecture.

Smith, P. A. Foundation of Lie groups. Ann. of Math. (2) 48, 29–42 (1947).

The purpose of the paper is to weaken further the conditions under which it can be asserted that a locally Euclidean topological local group G is a local Lie group (i.e., admits an analytical local coordinate system). It is proved that it is sufficient to postulate the existence of a left-regular coordinate system $(a^1, \cdots, a^r)$, i.e., of a coordinate system such that the coordinates of the product ab of two elements a and b near the identity can be expressed (vectorially) in the form $a+b+|a|F(a, b)$, where $F(a, b)$ tends to zero when a and b tend to zero ($|a|$ is the length of the vector a). Using results derived in a previous paper by the author [Ann. of Math. (2) 44, 481–513 (1943); these Rev. 5, 59] the existence first of one-parametric subgroups and then of a system of canonical coordinates is established. Denote by y_α the point with canonical coordinates $\alpha_1, \cdots, \alpha_r$; then it is proved that we can write $y_\alpha y_\beta = y_\gamma$, where the γ's are analytic functions of the α's and the β's. In order to do this, the functions γ are first constructed as solutions of a certain system of differential equations which can be written down because it is known that, in canonical coordinates, xyx^{-1} is linear in y and analytic in x. This being done, the proof that $y_\alpha y_\beta$ is actually equal to y_γ proceeds in two steps: it is first proved that $(y_\alpha y_\beta)x(y_\alpha y_\beta)^{-1} = y_\gamma x y_\gamma^{-1}$ for any x, i.e., that $y_\alpha y_\beta y_\gamma^{-1}$ is in the center of G; then, making use of the initial conditions for the functions γ, it is proved that $y_\alpha y_\beta = y_\gamma$. The paper also contains some minor corrections to the author's paper quoted above. *C. Chevalley.*

Andreoli, Giulio. Sulla teoria della sostituzioni generalizzate e dei loro gruppi generalizzati. Rend. Accad. Sci. Fis. Mat. Napoli (4) 10, 115–127 (1940).

The author generalizes the concept of a permutation

$$P = \begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$$

by also permitting that the first and second line of P contain equal elements. Multiplication of these "dispositions" is defined as for permutations and the associative law still holds. A system of such dispositions which is closed under multiplication is termed a generalized group. Several examples of generalized groups are given. *H. B. Mann.*

Richardson, A. R. Congruences in multiplicative systems. Proc. London Math. Soc. (2) 49, 195–210 (1946).

In a group there exists a structure isomorphism between the subgroup intersection and union and corresponding operations for the coset expansions considered as equivalence relations or partitions of the group. The author generally studies partitions defined by subsystems in a multiplicative system Σ. When A is a subset of Σ one writes $u \sim v \pmod{R_A}$ when there exists $a\epsilon A$ such that $u = va$. To make this an equivalence relation there must exist for every w elements in A such that the following three relations hold: (1) $w = wa$, (2) $w = (wa_1)a$, (3) $(wa_1)a_2 = wa$. A set A satisfying these conditions is called an E-set.

When B^A denotes the set of elements b in B for which $wb = wa$ for all b and some a in A then one finds $B \cap b = B^b \cap b^B$. Furthermore, for E-sets, B^A and the intersection $A \cap B$ are E-sets. There exist a maximal set $X = A_M$ and a minimal set $Y = A_m$ such that $A^X = A$. For the corresponding partitions one has $R_{A_m} = R_A = R_{A_M}$ and all B with the same partition as A belong to the same quotient system A_M/A_m and the correspondence $R_A \rightarrow A_M/A_m$ is a cross-cut isomorphism.

When the E-sets are closed with respect to multiplication one can establish that the union $A \cup B$ is also an E-set provided certain weak associative laws, used previously in a paper by Murdoch and Ore [Amer. J. Math. 63, 73–86 (1941); these Rev. 2, 245], shall hold. Then one arrives at the basic result that the correspondence $R_A \rightarrow A_M/A_m$ is a structure isomorphism. Finally one finds a discussion of the application of certain concepts for equivalence relations introduced by Jacotin-Dubreil. *O. Ore.*

Kuntzmann, J. Contribution à l'étude des systèmes multiformes. Ann. Fac. Sci. Univ. Toulouse (4) 3, 155–194 (1939).

This paper deals with several topics connected with multivalued multiplication. A "multiform system" M is a set of elements with an operation such that the product of two elements is a subset of the system. If no product is the null

set then the operation is said to be "perfect." The division symbol $a\lfloor\underline{b}$ will designate the set of all x such that $a\varepsilon bx$. Under this operation the system is also a multiform system. The symbol $\underline{a}\rfloor b$ is similarly the set of solutions of $b\varepsilon xa$. If $a\lfloor\underline{c}\approx\underline{b}\rfloor d$ implies that $ab\approx cd$ then the system is called "normal." [The condition is here restated in the notation of Prenowitz, $S\approx T$ meaning that the intersection is not void.] If the converse holds, then it is said to be "resolvable."

A "multigroup" is defined to be a multiform system for which multiplication and both divisions are perfect. A "hypergroup" is a multigroup which is associative and has a scalar unity e, $ex=x=xe$, with unique inverses, $\underline{a}\rfloor e=\bar{a}=e\lfloor\underline{a}$. Several examples are given, including that of functions $f(x, y)$ of two variables with the composition $f\cdot g$ equal to the Sylvester resultant of $f(x, z)$ and $g(z, y)$. A derived multiform system can be built up by dividing M into classes and defining a new operation in terms of the original. To do this one supposes there is a class of elements C_a for each element a. If $C_a\supset b$ implies $C_a\supset C_b$ then the division is called "regular." If there are two divisions such that $C_a\supset C_a{}^1$ for each a then C is said to be "consecutive" to C^1. Divisions are composed in two ways:

$$(CC')_a=C_{C'_a}\cup C_a\cup C'_a$$

or

$$(C\cdot C')_a=C'_a\cup C_{C'_a}\cup C'_{C_{C'_a}}\cup\cdots.$$

The second of these is distinguished by the name "strong." It is commutative, $C\cdot C^1=C^1\cdot C$. A division is regular if and only if $CC=C$. The division $C\cdot C$ is the least regular division consecutive to C. Composition and intersection among divisions are shown to satisfy the Dedekind condition, $A\supset C$ implies $(CB)\cap A=C(B\cap A)$.

For any division into classes C there exists another, N, the class Na consisting of all elements b such that $C_b=C_a$. A division can lead to several new multiform systems depending on how multiplication is defined. One way is to define $a\cdot b\supset c$ if $C_aC_b\supset c^1$ such that $c^1\varepsilon C_c$. Another would have $a\cdot b\supset c$ if $N_aN_b\supset c^1$ such that $c^1\varepsilon N_c$. Yet another would be for $a\cdot b\supset c$ if $C_aN_b\supset c^1$ such that $c^1\varepsilon N_c$. There are eight ways in all, called "symorphisms" of types CCC, NNN, CNN, etc.

The remainder of the paper consists of numerous comments on the relations among the concepts introduced above and homomorphisms, semi-homomorphisms, and so forth.

H. Campaigne (Arlington, Va.).

Climescu, Al. C. Sur l'équation fonctionnelle de l'associativité. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași] 1, 211–224 (1946).

From the study of the associative law one is led naturally to the functional equation $f(x, f(y, z))=f(f(x, y), z)$, where f is a single-valued function from $E\times E$ to E, the domain of definition. If $U(x)$ and $U^{-1}(x)$ are defined and single-valued from E to E, then any solution $f(x, y)$ of the functional equation will give another, $U^{-1}(f(u(x), u(y)))$. As a corollary to this, the associativity of multiplication among the complex numbers follows from that of addition. Complete solutions for several special cases are obtained. The problem of extending the definition of $f(x, y)$ from two points to the unit interval is solved completely. *H. Campaigne.*

NUMBER THEORY

***Ball, W. W. Rouse. Mathematical Recreations and Essays. Revised by H. S. M. Coxeter.** The Macmillan Company, New York, 1947. xvi+418 pp. $2.95.

This is a reprint of the eleventh edition [London, 1939]. It differs greatly from the tenth edition [1922, 1926]. Coxeter has raised the level of the discussion of a great many topics without losing the entertaining character of Ball's famous and familiar book. Most of the material added is of recent date and, as Coxeter puts it, is "material that Ball himself would have enjoyed." Three chapters of the tenth edition have been deleted; these dealt with Mechanical recreations, Bees and their cells, and String figures. The chapter on Cryptography has been completely rewritten by A. Sinkov. A chapter on Polyhedra has been added and four other chapters have been mostly rewritten. Topics like the coloring of maps and anallagmatic tesselations have been greatly strengthened. There are many new references to recent literature throughout the book. As well as for its entertainment value, the book is good reading for the young student who feels that all problems in elementary mathematics are either trivial or easily solved.

D. H. Lehmer (Berkeley, Calif.).

Finsler, P. Über die Faktorenzerlegung natürlicher Zahlen. Elemente der Math. 2, 1–11 (1947).

This paper discusses in detail a method of factoring (or proving the primality of) an integer by means of its terminal digits. In case the number n is small and no factor table is available a slide rule may be used. If $p<q$ are two candidates for the equation $pq=n$, then the last digit of n determines what last digit q must have when the last digit of p is given. For a given p, the slide rule gives at once the first digits of q. If the required last digit of q is annexed and the result is not a prime then the proposed p is impossible. In case of larger numbers the last two or even three digits may be considered and a more accurate slide rule employed. With this process the author is able to verify that 100 000 007 is a prime. Tables of reciprocals may also be used and finally computing machines. The method is most effective when dealing with numbers which are primitive factors of binomials like $a^{25}+b^{25}$ so that the last two digits of any factor are 01 or 51. On page 7 there is a table of the numerators and denominators of the first 20 convergents to π and their prime factors. *D. H. Lehmer* (Berkeley, Calif.).

Aubert, Karl E. Remark on the middle binomial coefficient. Norske Vid. Selsk. Forh., Trondhjem 17, no. 4, 13–16 (1944). (Norwegian)

Remark on the middle binomial coefficient in connection with former papers of J. E. Fjeldstad and W. Ljunggren [Norsk Mat. Tidsskr. 24, 13–17, 18–22 (1942); these Rev. 8, 314]. *T. Nagell* (Uppsala).

Kesava Menon, P. A generalization of Wilson's theorem. J. Indian Math. Soc. (N.S.) 9, 79–88 (1945).

Let P be the product of the kth power residues in a reduced system of residues mod m. Using the usual method for the proof of Wilson's theorem, the author determines the residue of P mod m. Most of the lemmas are well known. The main result is not new either, at least in the special cases $m=p^n$ and $m=2p^n$ [F. Arndt, J. Reine Angew. Math. 31, 333–342 (1846); P. Bachmann, Niedere Zahlentheorie, vol. 1, Leipzig, 1921, pp. 343–351]. *A. Brauer.*

Petr, K. Über die Bernoullischen Polynome. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 44, 511–526 (1943).

Polynomials $\phi_k(x)$ are defined by means of $\phi_k(x+1)-\phi_k(x)=x^k$, $\phi_k(0)=0$, where k is a positive integer. The author considers the representations

$$\phi_{2r-1}(x)=c_0^{(r)}\mathfrak{A}_r(x)+\cdots+c_{r-1}^{(r)}\mathfrak{A}_1(x),$$

$$\phi_{2r}(x)=d_0^{(r)}\mathfrak{B}_r(x)+\cdots+d_{r-1}^{(r)}\mathfrak{B}_1(x),$$

where

$$\mathfrak{A}_k(x)=\frac{1}{2k}\prod_{i=-k}^{k-1}(x+i),\quad \mathfrak{B}_k(x)=\frac{2x-1}{2(2k+1)}\prod_{i=-k}^{k-1}(x+i).$$

He proves (1) that the coefficients $c_j^{(i)}$, $d_j^{(i)}$ are positive integers, (2) $c_j^{(i)}=d_j^{(i)}$, $0\leqq j<i$, (3) $c_j^{(i)}=c_j^{(i-1)}+(i-j)^2c_{j-1}^{(i-1)}$, $c_0^{(i)}=d_0^{(i)}=1$. Various congruential properties of the $c_j^{(i)}$ are also proved and applied to the Bernoulli and related numbers. The paper closes with a proof of the Staudt-Clausen theorem. *L. Carlitz* (Durham, N. C.).

Levit, R. J. The non-existence of a certain type of odd perfect number. Bull. Amer. Math. Soc. 53, 392–396 (1947).

Let $\sigma(k)$ be the sum of the divisors of k. If the integer $n=p_1^{a_1}p_2^{a_2}\cdots p_t^{a_t}$ is perfect, then one and only one of the numbers $\sigma(p_\nu^{a_\nu})$ is even. Let us assume that $\sigma(p_1^{a_1})$ is even. We set $\sigma_1=\frac{1}{2}\sigma(p_1^{a_1})$, $\sigma_\nu=\sigma(p_\nu^{a_\nu})$, $\nu=2, 3, \cdots, t$. It is well known that for an even perfect number each σ_ν is a power of a prime. The author proves that an odd integer n cannot be perfect if each σ_ν is a power of a prime. *A. Brauer.*

Lehmer, D. H. On the factors of $2^n\pm1$. Bull. Amer. Math. Soc. 53, 164–167 (1947).

An investigation concerning the converse of Fermat's theorem disclosed that Kraitchik's table [Théorie des Nombres, v. 1, Paris, 1922] giving the least exponent e of 2 (mod p) contains numerous errors in the previously unchecked region $p>10^5$. Hence the author decided to make an independent examination of primes, considerably beyond 10^5, having small e. As a by-product numerous new factors of $2^n\pm1$ ($n<500$) were discovered; a one-page list of them is reproduced in the article. The methods used will be described elsewhere. It is believed that all factors less than 10^6 have now been found. Moreover any further factors of 2^n-1 for $n\leqq300$ and of 2^n+1 for $n\leqq150$ lie beyond 4538800. A list of eight new complete factorizations of numbers $2^n\pm1$ resulting from the above list is given. Eleven new factors pertain to Mersenne numbers, among which are factors of $2^{167}-1$ and $2^{229}-1$ which confirm their composite character, announced by Uhler and by Barker [Bull. Amer. Math. Soc. 52, 178 (1946); Math. Tables and Other Aids to Computation 2, 94 (1946); these Rev. 7, 273, 413]. The present state of our knowledge about Mersenne numbers is summarized. *N. G. W. H. Beeger* (Amsterdam).

Gloden, A. Table de factorisation des nombres x^4+1 dans l'intervalle $1000<x\leqslant3000$. Inst. Grand-Ducal Luxembourg. Sect. Sci. Nat. Phys. Math. Arch. N.S. 16, 71–88 (1946).

This table gives factors of x^4+1 ($(x^4+1)/2$ if x is odd) for the range $1001<x\leqq3000$. It contains 1417 complete factorizations. In 124 other cases a small prime factor is given and in 459 cases the table has a blank entry, indicating a number of unknown character. All unknown factors exceed 500000. The table is a by-product of the author's tables of solutions of the congruence $x^4+1\equiv0$ (mod p) [Mathematica, Timişoara 21, 45–65 (1945); these Rev. 7, 145]. *D. H. Lehmer* (Berkeley, Calif.).

***Gloden, A. Liste des Formes Linéaires des Nombres dont le Carré se Termine dans le Système Décimal par une Tranche Donnée de 4 Chiffres.** A. Gloden, Luxembourg, 1947. 14 pp.

This table gives all 1044 four-digit endings of squares arranged in increasing order. With each such ending is given the set of arithmetical progressions which contain those numbers whose squares end as specified. A classification of these linear forms is given in the introduction. *D. H. Lehmer* (Berkeley, Calif.).

***Gloden, Albert. Mehrgradige Gleichungen.** Mit einem Vorwort von Maurice Kraitchik. 2d ed. P. Noordhoff, Groningen, 1944. 104 pp. 2.50 florins; bound, 3.25 florins.

By a multigrade equation is meant a system of r equations

$$\sum_{\nu=1}^{p}a_\nu^{n_i}=\sum_{\nu=1}^{p}b_\nu^{n_i},\qquad i=1,\cdots,r,$$

in which the a's and b's are unknown integers and the n's are given nonnegative integers. The problem of finding solutions of such a system is also known as the Tarry-Escott problem and has an extensive literature. The purpose of this book is to set forth the various methods of solving the problem and to treat the several applications of its solutions. In the more important and interesting problems the given integers n_i are the consecutive integers from 1 to n inclusive, in which case the above system is frequently written

$$(1)\qquad a_1, a_2, \cdots, a_p \overset{n}{=} b_1, b_2, \cdots, b_p.$$

The contents of the eight chapters may be described briefly as follows. Chapters I and II contain definitions and general properties of solutions and their transformations. In case $n=p-1$ the equations (1) are called normal (ideal). Chapter III is devoted to such equations. The present state of this problem is as follows. For $n\leqq3$ the problem is completely solved. For $4\leqq n\leqq7$ parametric solutions are known. For $n=8$ only two essentially different numerical solutions are known, from which infinitely many examples with $n=9$ may be found. Chapter IV deals with various abnormal cases of $n\geqq10$ and the possible bounds for p. Chapters V and VI are concerned with other types of related Diophantine systems. Chapter VII treats the more general problem of finding more than two sets of integers having equal sums of like powers. These are called chains. For example, in each of the four sets (1, 17, 18), (2, 13, 21), (3, 11, 22), (6, 7, 23) the sum is 36 and the sum of squares is 614. The final chapter gives applications of multigrade equations. These include problems in the theory of equations, the calculation of logarithms and arccotangents of integers, irreducible polynomials and Waring's problem. Copious references to recent literature are given throughout the text. This work is a translation of a first edition in French which appeared in Luxembourg in 1938. *D. H. Lehmer.*

Gloden, A. Un théorème sur les multigrades. Inst. Grand-Ducal Luxembourg. Sect. Sci. Nat. Phys. Math. Arch. N.S. 16, 61–63 (1945).

This note has to do with the so-called Tarry-Escott problem [cf. the preceding review]. The theorem shows how to

find two sets of five integers having equal sums of mth powers for $m=1, 3, 5$ and 7 given six integers $a_1, a_2, a_3, b_1, b_2, b_3$ such that

$$a_1^2+a_2^2+a_3^2=b_1^2+b_2^2+b_3^2,$$
$$a_1^4+a_2^4+a_3^4=b_1^4+b_2^4+b_3^4,$$
$$a_1+a_2-a_3=2(b_1+b_2-b_3)\neq 0.$$

D. H. Lehmer (Berkeley, Calif.).

Dorwart, H. L. Sequences of ideal solutions in the Tarry-Escott problem. Bull. Amer. Math. Soc. **53**, 381–391 (1947).

For a statement of the problem and notations see the second preceding review. A solution (of degree k) is represented as

$$(1)\qquad a_1, a_2, \cdots, a_s \stackrel{k}{=} b_1, b_2, \cdots, b_s$$

and it implies $Ma_i+K \stackrel{k}{=} Mb_i+K$, $i=1, 2, \cdots, s$, and

$$(2)\quad a_1, a_2, \cdots, a_s, b_1+h, b_2+h, \cdots, b_s+h \stackrel{k+1}{=} b_1, b_2, \cdots, b_s, a_1+h, a_2+h, \cdots, a_s+h$$

with arbitrary M, K and h. The set of all the differences $|a_i-a_j|$, $|b_i-b_j|$ is called the difference table of (1); (1) is an ideal solution if $s=k+1$. If in its difference table a number h appears exactly h times, then (2) is reduced to an ideal solution (of degree $k+1$) that is said to be in sequence with the ideal solution of degree k. All sequences of ideal solutions beginning with $k=1, 2, 3$ are determined and arranged in a table. Certain substitution groups in the solutions are noted. No new solutions are given. [See also Dorwart and Brown, Amer. Math. Monthly **44**, 613–626 (1937); Chernick, ibid., 626–633 (1937); and Gloden's book reviewed above.] *N. G. W. H. Beeger* (Amsterdam).

Palamà, G. Contributo alla ricerca di soluzioni intere di sistemi indeterminati. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **6**(75), 437–452 (1942).

The author is concerned with finding integral solutions of a system of equations of the form

$$\sum_{j=1}^{n} u_j^{m_i} = \sum_{j=1}^{n'} v_j^{m_i}, \qquad i=1, 2, \cdots, r,$$

particularly for the cases where $n=n'$ and $m_i=i$, $2i$ or $2i-1$. Following a procedure due to Barbette [Association Française Avancement Sci. C. R. **49**, 93–96 (1925)] solutions can easily be obtained for sufficiently large n (e.g., $n=2^r$ will suffice in the case $m_i=i$). The author obtains solutions for $m_r\leqq 12$ for which n does not exceed 26. *F. A. Behrend.*

Bell, E. T. Diophantine equations suggested by elementary geometry. Ann. of Math. (2) **48**, 43–50 (1947).

The problem of finding complete solutions (i.e., solutions with exactly the right number of parameters) of even very simple looking Diophantine equations is a delicate task. The present paper gives such solutions for several types of equations which arise in elementary geometry. For example, the complete solution of $xyzw=t^2$, which is connected with the problem of finding triangles with integral sides and areas, contains ten rational integral parameters:

$$x=1\cdot a_{11}^2\cdot a_{12}a_{13}a_{14},\quad y=a_{12}\cdot a_{22}^2\cdot a_{23}a_{24},\quad z=a_{13}a_{23}\cdot a_{33}^2\cdot a_{34},$$
$$w=a_{14}a_{24}a_{34}\cdot a_{44}^2,\quad t=a_{11}a_{12}a_{13}a_{14}a_{22}a_{23}a_{24}a_{33}a_{34}a_{44}.$$

The general solution of $x_1x_2\cdots x_n=t^2$ is built similarly and contains $n(n+1)/2$ parameters.

As another example, the equation $\sum z_j^2=0$, $j=1, \cdots, n$, $z_j=x_j+iy_j$, x, y rational integers, is equivalent to the Diophantine system $\sum x_j^2=\sum y_j^2$, $\sum x_jy_j=0$, $j=1, \cdots, n$. For $n=4$ the general solution contains eight rational integral parameters: $2x_1=a_1b_1-a_2b_2+a_3b_3-a_4b_4$. For $x_2, x_3, x_4, y_1, y_2, y_3, y_4$ the a's occur in the same order as in x_1 and the indices of the b's and the signs of the summands of $2x_1, 2x_2, \cdots, 2y_4$ are, respectively, as follows: $(1, -2, 3, -4)$, $(2, 1, -4, -3)$, $(3, -4, -1, 2)$, $(4, 3, 2, 1)$, $(2, 1, 4, 3)$, $(-1, 2, 3, -4)$, $(4, 3, -2, -1)$, $(-3, 4, -1, 2)$. Misprints noticed: (2.1), in product read $r=1$ instead of $r=i$; (2.4), in x_4 read a_{21}^2 instead of a_{11}^2. *A. J. Kempner.*

Ljunggren, Wilhelm. On the Diophantine equation $x^2+p^2=y^n$. Norske Vid. Selsk. Forh., Trondhjem **16**, no. 8, 27–30 (1943).

Let p be a prime such that p^2-1 is exactly divisible by an odd power of 2. Then the equation of the title has only a finite number of solutions in x, y and n with $n>1$. The proof provides a method for finding all solutions.

I. Niven (Eugene, Ore.).

Ljunggren, Wilhelm. On the Diophantine equation $x^2+D=y^n$. Norske Vid. Selsk. Forh., Trondhjem **17**, no. 23, 93–96 (1944).

The author shows that the equation $x^2+D=y^n$ ($n>1$ and odd; $D\equiv 1 \pmod 4$, squarefree and positive) is not satisfied by any integers x and y if $D-1$ is exactly divisible by an odd power of 2, and if n does not divide the class number of the field $K\sqrt{(-D)}$. The method of proof is similar to that used by the author in an earlier paper [see the preceding review; see also the author's later paper in the same Forh. **18**, no. 32, 125–128 (1945); these Rev. **8**, 136].

R. A. Rankin (Cambridge, England).

Ljunggren, Wilhelm. Über die Gleichung $(Mx^2-N)^2=My^2-N$. Norske Vid. Selsk. Forh., Trondhjem **15**, no. 18, 67–70 (1942).

It was shown by E. Trost [Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 138–142 (1940); these Rev. **2**, 347] that in the case $M=N(N+1)$, $N=D^2$ $(D\neq 1)$ the equation $(Mx^2-N)^2=My^2-N$, where M and N are positive integers, has no nonnegative integral solutions except $x=0$, $y=1$ and $x=2$, $y=4N+1$. In the present paper it is shown that this result also holds for any integer $N>1$. In his proof, which depends upon unit theory, the author uses results which he obtained in an earlier paper [Skr. Norske Vid.-Akad. Oslo. I. **1936**, no. 12].

W. H. Gage (Vancouver, B. C.).

Ljunggren, Wilhelm. Über die Gleichungen $1+Dx^2=2y^n$ und $1+Dx^2=4y^n$. Norske Vid. Selsk. Forh., Trondhjem **15**, no. 30, 115–118 (1942).

If the number h of ideal classes in $R(\sqrt{(-D)})$ is not divisible by n, then the Diophantine equations of the title, with $D\equiv 1 \pmod 4$ and $D\equiv 3 \pmod 4$, respectively, have no solutions in integers x, y with $y>1$. *I. Niven.*

Ljunggren, Wilhelm. Solution complète de quelques équations du sixième degré à deux indéterminées. Arch. Math. Naturvid. **48**, no. 7, 35 pp. (1946).

Consider the Diophantine equation (1) $Ax^6-By^6=C$, where $A>0$, $B>0$, $(AB, C)=1$, AB is not a square or a cube and AB is not divisible by the 6th power of a prime. If $R((-27A/B)^{1/6})$ and $R((-27A_1/B_1)^{1/6})$ are identical fields, say that (1) and $A_1x^6-B_1y^6=C_1$ belong to the same family.

Then, for C restricted to 1, 2, 3, 4, 6 or 12, at most one equation in a family is solvable. If $C=1, 2, 3, 4$ or 6, equation (1) has at most one solution in positive integers; and if ξ is the fundamental unit of $R((-27A/B)^{1/6})$ of relative norm 1 with respect to the subfields $R((A/B)^{1/3})$ and $R((-3A/B)^{1/2})$, then ξ or ξ^2 is given by an explicit formula involving A, B, C and the solution of (1) if it exists (except in case $A=5$, $B=4$, $C=1$, when ξ^4 is so expressed). The proofs involve detailed discussions of units of the various fields. The methods are also applied to the equation $x^3+Dy^3=1$ which has been treated by B. Delaunay [C. R. Acad. Sci. Paris **162**, 150–151 (1916)] and T. Nagell [J. Math. Pures Appl. (9) **4**, 209–270 (1925)] and to the more general equation (2) $Ax^3+By^3=C$ which is discussed in Nagell's paper. The author states that he simplifies Nagell's procedure for solving (2). *I. Niven.*

Skolem, Th. Die Anzahl der Wurzeln der Kongruenz $x^3+ax+b\equiv 0 \pmod p$ für die verschiedenen Paare a, b. Norske Vid. Selsk. Forh., Trondhjem **14** (1941), no. 43, 161–164 (1942).

A table is set up giving, for fixed $a\neq 0$ and prime p, the number of values of b for which there are 0, 1, 2 or 3 solutions of the congruence of the title. The table is divided according as a is a quadratic residue or nonresidue of p and as $p\equiv 1$ or $2 \pmod 3$. A second tabulation gives the number of pairs (a, b) with $a\neq 0$ for which the congruence has 0, 1, 2 or 3 solutions. *I. Niven.*

Simmons, H. A. Classes of maximum numbers associated with symmetric equations in n reciprocals. IV. Proc. Phys.-Math. Soc. Japan (3) **23**, 687–695 (1941).

[Part III, by Simmons and Block, appeared in Duke Math. J. **2**, 317–339 (1936).] Let b, c, m and $n>1$ be arbitrary positive integers, and a_i, $i=1, \cdots, m$, any positive real numbers. The equation

$$\sum_{j=1}^{n} 1/x_j+\sum_{i=1}^{m} a_i\lambda^{-i}=b/a,$$

where $a=(c+1)b-1$ and λ is the product of the x_j, is discussed. Say that an E-solution of this equation is one such that $x_1\leqq\cdots\leqq x_n$, with all these except x_n required to be positive integers. One E-solution is

$$w_1=c+1,\quad w_{j+1}=aw_1w_2\cdots w_j+1,\quad j=1, \cdots, n-2,$$

and w_n determined by solving the equation for x_n. It is proved that the maximum λ among E-solutions is λ_w, which is the product of the w_j; that, among E-solutions for which (1) $\sum_{i=2}^{m} a_i\lambda^{1-i}<\sum_{i=2}^{m} a_i\lambda_w^{1-i}+1$, the largest value of x is w_n; that, under hypothesis (1), $P(x_j)<P(w_j)$, where $P(x_j)$ is any symmetric polynomial in the x_j, not a constant and with no negative coefficients. *I. Niven.*

***Ricci, Giovanni. Sull'irrazionalità del rapporto della circonferenza al diametro.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 147–151. Edizioni Cremonense, Rome, 1942.

Using a well-known method of Hermite the author gives simple proofs for the following two theorems. (I) If a, b and c denote rational integers, such that $(a+bi)/c\neq 0$, then it follows that $e^{(a+bi)/c}$ does not have the form $(A+Bi)/C$, where A, B and C also are rational integers. (II) For every positive integer m the number $e^{i\pi^m}$ is not real. On account of $e^{2iz}=(1+i\tan z)/(1-i\tan z)$ it follows from (I) that for every rational $p/q\neq 0$ the number $\tan p/q$ is irrational; hence π is irrational. As a corollary of (II) we obtain, on account of $e^{i\pi}=-1$, that π^2 is irrational. *J. Popken.*

Uspensky, J. V. On a problem of John Bernoulli. III. Revista Unión Mat. Argentina **11**, 239–255 (1946). (Spanish)

Uspensky, J. V. On a problem of John Bernoulli. IV. Revista Unión Mat. Argentina **12**, 10–19 (1946). (Spanish)

These articles continue the treatment of Bernoulli's problem on the sequence of greatest integers $[nx+b]$ [same Revista **11**, 141–154, 165–183 (1946); these Rev. **8**, 5, 6]. In part III, the intermediate convergents are studied in detail. They do not give best approximations if n is odd and only those of the form $(\mu P_n+P_{n-1})/(\mu Q_n+Q_{n-1})$ may be used if n is even. The case $b=\frac{1}{2}$ is investigated in great detail. A convenient arrangement for the practical construction of the Bernoulli series is tabulated in part IV, by use of the principal convergents and the intermediate ones of the form $M=\mu P_n+P_{n-1}$, $N=\mu Q_n+Q_{n-1}$. As an example, $[m(\sqrt{5}-1)/2]$ $(m=1, \cdots, 20)$ is constructed from the continued fraction expansion of $(\sqrt{5}-1)/2$; similarly $[m\sqrt{2}]$, and the minimal period of (13/56, 1/2). *G. Pall.*

Motzkin, Th. From among n conjugate algebraic integers, $n-1$ can be approximately given. Bull. Amer. Math. Soc. **53**, 156–162 (1947).

For any $n-1$ given complex numbers $z_1, \cdots, z_{n-1}$ and every $\epsilon>0$ there exists a system $\zeta_1, \cdots, \zeta_{n-1}$ of roots of an irreducible algebraic equation $\zeta^n+a_1\zeta^{n-1}+\cdots+a_n=0$ of degree n with complex integral coefficients such that $|\zeta_\nu-z_\nu|<\epsilon$ $(\nu=1, \cdots, n-1)$. The same is true for integral real coefficients, provided that the numbers $z_1, \cdots, z_n$ are symmetric with respect to the real axis. The author proves this theorem and adds some results on the position of the remaining root ζ_n in the complex plane. The proof uses the theorem of uniform continuity of the roots of an arbitrary algebraic equation as functions of the ratios of the coefficients (a topological proof of this theorem is given) and Kronecker's theorem on inhomogeneous linear Diophantine approximations. *J. F. Koksma* (Amsterdam).

Davenport, H. On the minimum of $x^3+y^3+z^3$. J. London Math. Soc. **21**, 82–86 (1946).

Let $m(k, n)$ (k, n are positive integers) denote the lower bound of the numbers λ satisfying the following conditions: if $x_1, \cdots, x_n$ are linear forms in $u_1, \cdots, u_n$ with real coefficients and determinant 1, then there exist integral values of $u_1, \cdots, u_n$ for which $0<|x_1^k+\cdots+x_n^k|\leqq\lambda$. Using a theorem of Blichfeldt [Trans. Amer. Math. Soc. **15**, 227–235 (1914)] in the geometry of numbers the author proves that $m(3, 3)\leqq 1.1571\cdots$. This is a better inequality than that of the author's recent note [J. London Math. Soc. **19**, 13–18 (1944); these Rev. **6**, 57]. The author shows that the same method is applicable to the investigation of the estimate of the number $m(k, n)$ in the general case, where n is a positive and k an odd positive integer. *V. Knichal* (Prague).

Davenport, H. On a theorem of Tschebotareff. J. London Math. Soc. **21**, 28–34 (1946).

Let $L_1, \cdots, L_n$ be real linear forms in $x_1, \cdots, x_n$ with determinant 1. Then it is known that for a certain ω_n, depending on n only, the inequality $|(L_1+c_1)\cdots(L_n+c_n)|\leqq\omega_n$ has integral solutions $(x_1, \cdots, x_n)$ for all sets of real constants

$(2e-1)^{-1}$

$c_1, \cdots, c_n$. Tschebotareff proved that $\omega_n \leqq (1+\epsilon)2^{-n/2}$ $(\epsilon>0)$, Mordell proved that $\omega_n \leqq \{(\sqrt{2})^n+(2-\sqrt{2})^n\}^{-1}$ and now the author deduces the improvement: $\omega_n \leqq \lambda_n 2^{-n/2}$, where $\lambda_n<1$ and $\lambda_n \to (1+2e)^{-1}$ as $n\to\infty$. [Minkowski's conjecture asserts that $\omega_n = 2^{-n}$ and has been proved for $n=2$ [Minkowski] and $n=3$ [Remak].] *J. F. Koksma.*

cf. J. London Math. Soc. 24, 316 (1949)

Davenport, H. On the product of three non-homogeneous linear forms. Proc. Cambridge Philos. Soc. 43, 137–152 (1947).

Let ξ, η, ζ be linear forms in u, v, w with real coefficients and determinant $\Delta\neq 0$. Minkowski's conjecture, proved by Remak [and afterward in a simpler way by Davenport, J. London Math. Soc. 14, 47–51 (1939)], states that for any real numbers a, b, c there exist lattice points (u, v, w) such that $|(\xi-a)(\eta-b)(\zeta-c)| \leqq \frac{1}{8}|\Delta|$. By a modification of his method, the author proves that, if none of the forms represents zero for any lattice point $(u, v, w)\neq(0, 0, 0)$, there exists a number M, depending only on the coefficients of the forms and satisfying $M<\frac{1}{8}$, such that for any real a, b, c there exist lattice points (u, v, w) for which

$$|(\xi-a)(\eta-b)(\zeta-c)| \leqq M|\Delta|.$$

The author investigates the "true value" of M (i.e., the lower bound $\bar{M}$ of all such numbers M, which turns out to be a number M itself) for two special sets of linear forms (they are also the forms for which the minimum of $|\xi\eta\zeta|/|\Delta|$ has its greatest possible values): (a) $\xi=\theta u+\varphi v+\psi w$, $\eta=\varphi u+\psi v+\theta w$, $\zeta=\psi u+\theta v+\varphi w$, where θ, φ, ψ are the roots of $t^3+t^2-2t-1=0$, the determinant of these forms being 7; the author proves that $\bar{M}=1/49$; (b) $\xi=u+\theta' v+\varphi' w$, $\eta=u+\varphi' v+\psi' w$, $\zeta=u+\psi' v+\theta' w$, where θ', φ', ψ' are the roots of $t^3-3t-1=0$, the determinant of these forms being 9; the author proves $\bar{M}=1/27$. *J. F. Koksma.*

Davenport, H. Non-homogeneous binary quadratic forms. Nederl. Akad. Wetensch., Proc. 49, 815–821 = Indagationes Math. 8, 518–524 (1946).

If $\alpha, \beta, \gamma, \delta$ are real numbers with $\Delta=\alpha\delta-\beta\gamma\neq 0$, then the theorem of Minkowski on nonhomogeneous linear forms asserts that for any two real numbers λ, μ there exists a lattice point (x, y) such that

$$|(\alpha x+\beta y+\lambda)(\gamma x+\delta y+\mu)| \leqq \tfrac{1}{4}|\Delta|.$$

This theorem can be expressed as a property of the indefinite binary form $ax^2+bxy+cy^2=(\alpha x+\beta y)(\gamma x+\delta y)=f(x, y)$. The author deduces the following theorem which is a little better than the theorem quoted above. For any indefinite quadratic form $f(x, y)$ which does not represent zero for any lattice point other than $(0, 0)$, there exists a number $M<\frac{1}{4}$, such that for any two numbers x_0, y_0 there exist x, y with $x\equiv x_0 \pmod 1$, $y\equiv y_0 \pmod 1$, such that $|f(x, y)| \leqq Md^{\frac{1}{2}}$, $d=b^2-4ac=\Delta^2$. The author defines $M(f)$ to be the lower bound of all such numbers M and investigates its properties. First he deduces an estimate $M(f)$ in terms of any value of f which corresponds to coprime integral values of x, y and which satisfies $0<f_1<d^{\frac{1}{2}}$. Furthermore, he proves that, if $f(x, y)=x^2+2kxy-y^2$, where k is a positive integer, then $M(f)=\frac{1}{4}k(k^2+1)^{-\frac{1}{2}}$ which includes the assertion that the inequality $M(f)<\frac{1}{4}$ is the best possible general inequality for $M(f)$. Finally the author proves that for any form f of Markoff's eries [see, for example, Bachmann, Die Arithmetik der quadratischen Formen, vol. 2, Teubner, Leipzig, 1923, chap. 4] we have $M(f)<\frac{1}{4}(5/9)^{\frac{1}{2}}$. *J. F. Koksma.*

Khintchine, A. Sur le problème de Tchebycheff. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 281–294 (1946). (Russian. French summary)

Let θ be an irrational real number, α a real number such that $\theta x-y-\alpha\neq 0$ for integral x, y. Denote by $\lambda(\theta, \alpha)$ the lower bound of the positive numbers C for which $|\theta x-y-\alpha|<C/|x|$ is soluble in integers x, y with arbitrarily large $|x|$; by λ the number $\lambda(\theta, 0)$; and by $\mu(\theta)$ the upper bound of $\lambda(\theta, \alpha)$ for all α satisfying the condition above. A classical theorem of Minkowski [for the literature, see Koksma, Diophantische Approximationen, Ergebnisse der Math., v. 4, no. 4, Springer, Berlin, 1936, chap. 6] states that $\mu(\theta)\leqq\frac{1}{4}$; the author improves this inequality to $\mu(\theta)\leqq\frac{1}{4}(1-4\lambda^2)^{\frac{1}{2}}$; he shows that the right-hand side cannot be replaced by a smaller analytic function of λ, and that the equality sign holds for

$$\theta=\frac{1}{k+}\frac{1}{k+}\frac{1}{k+}\cdots$$

when $k\geqq 1$ is an even integer, but not when it is an odd integer. [Compare also the paper of Davenport reviewed above.] *K. Mahler* (Manchester).

Rado, R. A theorem on the geometry of numbers. J. London Math. Soc. 21, 34–47 (1946).

Der Verfasser gibt eine Verallgemeinerung des Minkowskischen Satzes, der auch die Verallgemeinerungen von Mordell [Compositio Math. 1, 248–253 (1934)] und van der Corput [Acta Arith. 1, 62–66 (1935)] noch als Spezialfälle umfasst. Es seien $x, y, \cdots$ Punkte des n-dimensionalen Euklidischen Raumes; ξ durchlaufe ein Gitter mit der Determinante D; es enthalte den Nullpunkt. Es sei λ eine (n, n)-reihige Matrix. Eine Punktmenge S heisse eine λ-Menge, wenn mit x und y jedesmal auch $\lambda(x-y)$ der Menge angehört. Eine Funktion $f(x)$ heisse eine λ-Funktion, wenn für jede Zahl a die Menge derjenigen Punkte, für die $f(x)\geqq a$, eine λ-Menge bildet. Ist insbesondere $\lambda=\frac{1}{2}E$ gleich der halben Einheitsmatrix, so ist jede λ-Menge konvex und symmetrisch in bezug auf den Nullpunkt, und umgekehrt. Der Verfasser beweist den folgenden Satz. Ist $f(x)$ eine λ-Funktion, S eine λ-Menge und bedeutet $\sum_S'$ die Summierung über alle der Menge S angehörigen Gitterpunkte mit Ausnahme des Nullpunktes, so ist

$$f(0)+\tfrac{1}{2}\textstyle\sum_S' f(\xi) \geqq |D|^{-1}\|\lambda\| \displaystyle\int_S f(x)dx.$$

Setzt man $f(x)\equiv 1$ und $\lambda=\frac{1}{2}E$, so erhält man den Minkowskischen Satz. *O.-H. Keller* (Dresden).

Mahler, Kurt. The theorem of Minkowski-Hlawka. Duke Math. J. 13, 611–621 (1946).

Il résulte d'un théorème de géométrie intégrale, dû à C. L. Siegel [Ann. of Math. (2) 46, 340–347 (1945); ces Rev. 6, 257] que, si K est un corps de volume non supérieur à 1 dans R^n, il existe un réseau unitaire (i.e., transformé, par une substitution unimodulaire réelle, du réseau des points à coordonnées entières) dont aucun point, sauf peut-être O, ne soit intérieur à K; si K est étoilé symétrique de centre O, il résulte du même théorème qu'il suffit de supposer K de volume non supérieur à $2\zeta(n)$ (c'est le théorème "de Minkowski-Hlawka"). L'auteur montre que, si K est convexe symétrique de centre O, la constante $2\zeta(n)$ n'est pas la meilleure possible. Sa démonstration (indépendante de celles de Siegel et de Hlawka) procède par récurrence sur n, et repose (a) sur une démonstration élémentaire directe du

résultat pour $n=2$; (b) sur un lemme applicable à tout corps K symétrique de centre O [sur ce lemme, cf. K. Mahler, J. London Math. Soc. **19**, 201–205 (1944); ces Rev. **7**, 244]; (c) sur le théorème de Brunn-Minkowski.
A. Weil (São Paulo).

Mahler, K. On irreducible convex domains. Nederl. Akad. Wetensch., Proc. **50**, 98–107 = Indagationes Math. **9**, 73–82 (1947).

Let K be a plane convex domain which is symmetric with respect to the origin. A lattice whose only point interior to K is the origin is known as K-admissible. Let $\Delta(K)$ denote the lower bound of the determinants of all the K-admissible lattices and let H be a star domain, symmetric with respect to the origin, with $H<K$. If, for every such H, $\Delta(H)<\Delta(K)$, K is defined to be irreducible. The purpose of this paper is to show that an irreducible convex domain H exists within every domain K with $\Delta(H)=\Delta(K)$. Using general results he established in an earlier paper [same Proc. **49**, 331–343 = Indagationes Math. **8**, 200–212 (1946); these Rev. **8**, 12], the author shows that every reducible convex domain K contains a smaller convex domain H with $\Delta(H)=\Delta(K)$ and then expresses the required irreducible convex domain by means of a convergent sequence of the smaller domains H.
D. Derry.

Mahler, K. On the area and the densest packing of convex domains. Nederl. Akad. Wetensch., Proc. **50**, 108–118 = Indagationes Math. **9**, 83–93 (1947).

If K and $\Delta(K)$ are defined as in the paper reviewed above, let $V(K)$ denote the area of K. The author studies the lower bound Q of $V(K)/\Delta(K)$ over all possible K. A domain K with $V(K)/\Delta(K)=Q$ is defined to be an extreme domain. The existence of extreme domains is established and they are proved to be irreducible. The determination of these domains is reduced to a problem in the calculus of variations. However, the integral of the resulting Euler equations, an ellipse, is shown not to be an extreme domain. An alternative formulation of the problem of the determination of the constant Q is given in terms of sets of nonoverlapping domains congruent to K whose centers form a lattice.
D. Derry (Vancouver, B. C.).

Venkataraman, C. S. A new identical equation for multiplicative functions of two arguments and its applications to Ramanujan's sum $C_M(N)$. Proc. Indian Acad. Sci., Sect. A. **24**, 518–529 (1946).

Using definitions and notations of Vaidyanathaswamy's theory of multiplicative functions [Trans. Amer. Math. Soc. **33**, 579–662 (1931)], the author gives the following modification of Vaidyanathaswamy's "identical equation":

$$F(M,N)=F(M,1)F(1,N)\cdot C(M,N).$$

By this equation (the dot indicates "composition") a "cardinal component" C of F is defined. Some properties of Ramanujan's sum $C_M(N)$ are derived from the identical equation for that function: $C_M(N)=E^{-1}(M)E(N)\cdot\text{princ }I$.
N. G. de Bruijn (Delft).

Lahiri, D. B. On a type of series involving the partition function with applications to certain congruence relations. Bull. Calcutta Math. Soc. **38**, 125–132 (1946).

Let $\delta(m)$ be defined to be $(-1)^s$ in case m is the pentagonal number $(3s\pm1)s/2$ and zero for nonpentagonal numbers. The author considers the (finite) sum

$$S_r(n)=\sum_{m=1}^{\infty}\delta(m)m^r p(n-m)$$

in which $p(k)$ denotes the number of unrestricted partitions of k. For $r=1, 2, 3, 4$ and 5 the author finds that $S_r(n)$ can be expressed as a linear combination of $\sigma_1(n), \sigma_3(n), \cdots, \sigma_{2r-1}(n)$ with coefficients which are polynomials in n. Here $\sigma_k(n)$ denotes the sum of the kth powers of the divisors of n. For example, $S_1(n)=\sigma_1(n)$ (a fact discovered by Euler), $12S_2(n)=(18n-1)\sigma_1(n)-5\sigma_3(n)$. These sums are used to obtain congruences (mod a) for the sums

$$I_{a,b}=\sum_{m\equiv b\,(\mathrm{mod}\,a)}^{\infty}\delta(m)p(n-m),\qquad b=0, 1, \cdots, a-1,$$

for $a=2, 3, 4, 5, 7, 11$. These are expressed in terms of $\sigma_r(n)$ and used to derive the congruences of Ramanujan,

$$p(5m+4)\equiv0 \pmod 5,\quad p(7m+5)\equiv0 \pmod 7,$$
$$p(11m+6)\equiv0 \pmod{11}.$$

The methods are based on papers 18 and 30 of Ramanujan's Collected Papers. For $r>5$, it appears necessary to use other functions than $\sigma_r(n)$.
D. H. Lehmer.

Bambah, R. P. Two congruence properties of Ramanujan's function $\tau(n)$. J. London Math. Soc. **21**, 91–93 (1946).

Proofs are given of the congruences $\tau(2k+1)\equiv\sigma_3(2k+1)$ (mod 32), $\tau(n)\equiv n\sigma_3(n)$ (mod 25), where $\sigma_k(m)$ denotes the sum of the kth powers of the divisors of m and $\tau(n)$ is the coefficient of x^{n-1} in the expansion of the 24th power of the infinite product $(1-x)(1-x^2)(1-x^3)\cdots$. It follows that $\tau(n)$ is divisible by 32 and 25 for almost all n. The method is based on certain identities of Ramanujan.
D. H. Lehmer (Berkeley, Calif.).

Nasir, Abdur Rahman. On a certain arithmetic function. Bull. Calcutta Math. Soc. **38**, 140 (1946).

The author considers the function $L(n)$ defined as the sum of all the divisors of n except 1 and n so that $L(p)=0$ if and only if p is a prime. The rth iterate of $L(n)$ is denoted by $L_r(n)=L(L_{r-1}(n))$; $L_1(n)=L(n)$. It is conjectured that for each n the sequence of iterates $L_r(n)$ assumes only a finite number of different values. This is verified for $n\leqq100$. The pair (48, 75) behave like amicable numbers, that is, $L(48)=75$ and $L(75)=48$, so that this gives a periodic sequence of iterates. It is conjectured that arbitrarily long sequences exist.
D. H. Lehmer (Berkeley, Calif.).

Bulat, P. M. On asymptotic estimates of the average values of a fundamental function of the additive theory of numbers. Bull. [Izvestiya] Math. Mech. Inst. Univ. Tomsk **3**, 104–110 (1946). (Russian)

If $X=\{x_i\}$ and $Y=\{y_j\}$ $(1\leqq i\leqq k_1, 1\leqq j\leqq k_2)$ are any two sequences of integers lying in the interval $(1, n)$, the function $f=f(x_1, \cdots, x_{k_1}; y_1, \cdots, y_{k_2}|n)$ is defined to be the number of integers in this interval which are expressible as an x_i, a y_j or as a sum x_i+y_j. The author estimates the average values of f when k_1 and k_2 are fixed and of various orders of magnitude, in the three cases when (i) the sequences X and Y are varied in all possible ways, (ii) X and Y are identical and are varied in all possible ways and (iii) Y is fixed and X is varied in all possible ways. Explicit expressions for these averages in terms of binomial coefficients were given by N. P. Romanov [same Bull. **1**, 190–204 (1937)] and form the starting point of the author's work.
R. A. Rankin (Cambridge, England).

Romanoff, N. P. Über die Bestimmung der höherer Mittelwerte der Fundamentalfunktion der additiven Zahlentheorie. Bull. [Izvestiya] Math. Mech. Inst. Univ. Tomsk 3, 128–144 (1946). (Russian. German summary)

This paper is concerned with the estimation of the mean value of the cube of the function f defined in the preceding review, the sequences X and Y being varied in all possible ways. For this purpose the sum $\sum_X \sum_Y f^p$ is regarded as the coefficient of $u^{k_1}v^{k_2}$ in a double power series $\Phi_{n,p}(u, v)$. Lengthy and complicated formulae involving several auxiliary functions are obtained by means of which $\Phi_{n,3}(u, v)$ and other similar functions may be derived. The author makes considerable use of results obtained by him in an earlier paper [same Bull. 2, 13–37 (1938)] which was concerned with the estimation of the mean value of f^2.

R. A. Rankin (Cambridge, England).

Romanov, N. P. The application of functional analysis to questions of the distribution of prime numbers. Bull. [Izvestiya] Math. Mech. Inst. Univ. Tomsk 3, 145–173 (1946). (Russian)

The author considers a set of distributive functional operators L_u $(u>0)$ which map a family of functions onto itself and which possess the multiplicative property $L_m L_n = L_{mn}$ for all m, n. This property is possessed, for example, by the operator

$$L_u = u^{-\lambda} = \sum_{r=0}^{\infty} \frac{(-\log u)^r}{r!} \lambda^r,$$

where λ is an arbitrary linear operator. It is shown that many of the properties of the Riemann zeta-function hold also for the operator $\zeta(s+\lambda) = \sum_{n=1}^{\infty} n^{-s} n^{-\lambda}$ when certain convergence conditions are satisfied and when the expressions obtained are interpreted in a suitable and natural manner. Thus $\zeta(s+\lambda)$ can be continued to the left of its region of absolute convergence and satisfies a functional relation connecting it with $\zeta(1-s-\lambda)$. This relation only differs from that which holds for the ordinary zeta-function in the presence of certain simple operators which vanish when λ is a pure number. The occurrence of these extra operators can be attributed to the fact that, in the theory of the Riemann zeta-function, the term $1/(s-1)$ due to the pole at $s=1$ appears first, for $\Re s>1$, as an integral $\frac{1}{2}\int_0^1 t^{\frac{1}{2}(s-1)-1} dt$ and is later identified with the integral $-\frac{1}{2}\int_0^1 t^{\frac{1}{2}(1-s)-1} dt$ when $\Re s<1$. This identification between the two integrals is not always possible in the case of integrals which are operators.

R. A. Rankin.

✶Brun, Viggo. La somme des facteurs de Möbius. C. R. Dixième Congrès Math. Scandinaves 1946, pp. 40–53. Jul. Gjellerups Forlag, Copenhagen, 1947.

This lecture contains a number of interesting remarks about the sum $M(x) = \sum_{n \leq x} \mu(n)$ and about associated questions.

H. Davenport (London).

Zahlen, Jean Pierre. Sur la repartition des nombres premiers relatifs dans certaines progressions arithmétiques et quelques problèmes connexes sur les nombres premiers absolus. Euclides, Madrid 6, 609–613 (1946).

Exposition of several curious well-known results.

H. S. Zuckerman (Seattle, Wash.).

Selmer, Ernst S., und Nesheim, Gunnar. Tafel der Zwillingsprimzahlen bis 200.000. Norske Vid. Selsk. Forh., Trondhjem 15, no. 25, 95–98 (1942).

This table lists those values of n for which both $6n+1$ and $6n-1$ are primes under 200000. The numbers of prime pairs in the two first 100000 numbers are thus found to be 1224 and 936, respectively.

D. H. Lehmer.

Selmer, Ernst S., und Nesheim, Gunnar. Die Goldbachschen Zwillingsdarstellungen der durch 6 teilbaren Zahlen 196.302–196.596. Norske Vid. Selsk. Forh., Trondhjem 15, no. 28, 107–110 (1942).

A pair of twin representations of an even number $2n$ as the sum of two odd primes is a set of two representations $2n = p_1 + q_1 = p_2 + q_2$ in which p_1 and p_2 differ by 2. Necessarily n is a multiple of three. In counting such representations the convention adopted counts two for each such pair unless p_1 and q_1 differ by two so that $p_1 = q_2$; in this case we count this as only one representation. With this convention Stäckel has deduced by probabilistic reasoning that the number of such representations of $6m$ is approximately

$$G(6m) = 4.1532\{\pi(6m)\}^4(6m)^{-3}\prod(p-2)/(p-4)\prod(q-3)/(q-4),$$

where p ranges over the prime factors greater than 3 of m, while q ranges over the odd prime factors of $3m \pm 1$. The object of this note is to compare this formula with the actual values of G for the 50 multiples of 6 between 196301 and 196597. Both approximate and exact values and also their ratio are given. The agreement is good. This agreement is also indicated graphically.

D. H. Lehmer.

✶Selberg, Sigmund. An asymptotic formula for the distribution of the two-factorial integers. C. R. Dixième Congrès Math. Scandinaves 1946, pp. 59–64. Jul. Gjellerups Forlag, Copenhagen, 1947.

Let $\pi_2(x)$ be the number of integers not exceeding x which can be represented as a product of two different primes. The author proves the formula

$$\pi_2(x) = x \sum_{n=1}^{r-1} (n! \log\log x + a_n)(\log x)^{-n-1} + O(x(\log x)^{-r-1}),$$

where r is a positive integer and $a_1, a_2, \cdots$ are real constants. The formula was given by Shah [Indian Phys. Math. J. 4, 47–53 (1933)] but the author states that Shah's proof was not correct.* The author proves the formula in the classical way by an investigation of the integral $\int x^s s^{-1} \log^2 \zeta(s) ds$. He also states that the formula can be proved by elementary methods.

H. Heilbronn.

Selberg, Atle. On the zeros of the zeta-function of Riemann. Norske Vid. Selsk. Forh., Trondhjem 15, no. 16, 59–62 (1942).

Announcement of results which have already been reviewed. Cf. Arch. Math. Naturvid. 45, no. 9, 101–114 (1942); these Rev. 7, 48.

✶Selberg, Atle. The zeta-function and the Riemann hypothesis. C. R. Dixième Congrès Math. Scandinaves 1946, pp. 187–200. Jul. Gjellerups Forlag, Copenhagen, 1947.

"The object of the following lecture is . . . to give an outline of some recent investigations in the distribution of the zeros of the zeta-function in relation to the critical line. Particular stress is laid upon explaining the main ideas underlying the methods." It is of interest that, in spite of the author's advance in proving the zeros on the critical line to be within a multiplicative constant of the total number in the critical strip, the author is suspicious of the Riemann hypothesis. He presents considerable evidence for being suspicious. A selected list of papers published since

* The error was corrected by Shah [Indian Phys. Math. J. 6, 19–25 (1935)]

Titchmarsh's tract [The Zeta-function of Riemann, Cambridge University Press, 1930] is appended.
N. Levinson (Cambridge, Mass.).

Guinand, A. P. Some Fourier transforms in prime-number theory. Quart. J. Math., Oxford Ser. **18**, 53–64 (1947).

Assuming the truth of the Riemann hypothesis, the author constructs two pairs of Fourier cosine-transforms illustrating the relationship between the nontrivial zeros $\frac{1}{2}+i\gamma_n$ of $\zeta(s)$ and the logarithms of the powers of the primes. [Cf. Wintner, Duke Math. J. **10**, 99–105 (1943); these Rev. **4**, 217.] In one of the examples, one function involves $\sum_{0<\gamma_n\leqq x} 1/\gamma_n$ and elementary functions, while its transform involves $\sum_{m\log p\leqq x} \log p/p^{\frac{1}{2}m}$ and elementary functions.
P. Hartman (Baltimore, Md.).

ANALYSIS

***Jeffreys, Harold, and Jeffreys, Bertha Swirles. Methods of Mathematical Physics.** Cambridge, at the University Press; New York, The Macmillan Company, 1946. ix+679 pp. $15.00.

The spirit of this book is indicated by the following extracts from the preface. "This book is intended to provide an account of those parts of pure mathematics that are most frequently needed in physics. . . . We do not accept the common view that any argument is good enough if it is intended to be used by scientists. We hold that it is as necessary to science as to pure mathematics that the fundamental principles should be clearly stated and that the conclusions shall follow from them. But in science it is also necessary that the principles taken as fundamental shall be as closely related to observation as possible. . . . We maintain therefore that careful analysis is more important in science than in pure mathematics, not less. We have also found repeatedly that the easiest way to make a statement reasonably plausible is to give a rigorous proof. . . . We consider it especially important that scientists should have reasonably accessible statements of conditions for the truth of the theorems that they use. . . . We have . . . often given proofs under more general conditions than are usually taught to scientists, where the usual sufficient conditions are often not satisfied in practice but less stringent ones are satisfied." In general, the authors have adhered to the standards which they have set for themselves. For example, careful attention is paid to questions of uniform convergence, and the authors give Schwarz's example to show that the area of a surface cannot be defined as the limit of the areas of inscribed polyhedra.

Concerning the choice of material, the authors state: "We have generally included a method if it has applications in at least two branches" of physics. However, some methods (e.g., Green's functions) are omitted as demanding too much space for adequate treatment. The authors have apparently not hesitated to discard traditional topics if they feel that they are no longer of sufficient importance or if they regard more modern methods as more satisfactory; this presumably accounts for their devoting only a page to integral equations. The numerous illustrative examples are in many cases drawn from quite recent literature and are frequently both more difficult and less hackneyed than those usually appearing in textbooks.

The following list of chapter headings gives a good idea of the topics covered: The real variable, Scalars and vectors, Tensors, Matrices, Multiple integrals, Potential theory, Operational methods, Physical applications of the operational method, Numerical methods, Calculus of variations, Functions of a complex variable, Contour integration and Bromwich's integral, Conformal representation, Fourier's theorem, The factorial and related functions, Solution of linear differential equations of the second order, The equations of potential waves, and heat conduction, Waves in one dimension and waves with spherical symmetry, Conduction of heat in one and three dimensions, Bessel functions, Applications of Bessel functions, The confluent hypergeometric function, Legendre functions and associated functions, Elliptic functions. Even the pure mathematician will find the book useful as a source of material not readily available in other books, such as the methods of steepest descent and stationary phase, and special conformal mappings.
R. P. Boas, Jr. (Providence, R. I.).

Popoviciu, Tiberiu. Notes sur les fonctions convexes d'ordre supérieur. X. Sur quelques propriétés des différences divisées et des polynomes de Lagrange. Ann. Sci. Univ. Jassy. Sect. I. **28**, 161–207 (1942).

[For note IX see Bull. Math. Soc. Roumaine Sci. **43**, 85–141 (1941); these Rev. **7**, 116.] A real function $F(x)$, defined on a linear set E, is said to be nonconcave of order k if the divided difference $[x_1, x_2, \cdots, x_{k+2}; f]\geqq 0$ for every set of $k+2$ distinct points of E. The author studies the relations between $f(x)$ and the polynomial $L(x_1, \cdots, x_{n+1}; f|x)$ of degree at most n interpolating $f(x)$ at the points $x_1, \cdots, x_{n+1}$ of E. It is shown that if E is a finite set composed of exactly m points ($m\geqq n+1$) there is always an interpolating polynomial dominating the function, i.e., such that $L(x_1, \cdots, x_{n+1}; f|x)\geqq f(x)$ for $x\epsilon E$. Again, for E finite and $f(x)$ nonconcave of order k it is shown that there always exists an interpolating polynomial of given degree n $(>k+1)$ which is also nonconcave of order k on the set E, provided k and n are of different parities. However, if k and n are of the same parity, this property need no longer hold, being now dependent on the geometric configuration of the set E. This point is studied in detail for $k=0$, $n=2$, $m=4$, and also for some other cases with $k>0$. The paper concludes with a discussion of dominating interpolating polynomials for the case when E is a finite closed interval with suitable assumptions on $f(x)$. Here the author uses the polynomials of best approximation.
I. J. Schoenberg.

de Toledo Piza, Affonso P. Representative values of a distribution. Indices of dispersion. Anais Acad. Brasil. Ci. **28**, 209–235 (1946). (Portuguese)

A representative value (or homogeneous internal mean) is a symmetric continuous function $M=f(X_1, \cdots, X_n)$ of m real variables such that $\min_i X_i\leqq f(X_1, \cdots, X_n)\leqq \max_i X_i$. If $\varphi(x, M)$ is continuous, differentiable in x, together with its inverse, and if $\varphi(M, M)=0$, then its inverse is a dispersing function. If $M=f(\varphi(x_1, M), \cdots, \varphi(x_m, M))$ is equivalent to an equation in the x_i not involving M, the latter equation is a characteristic equation for M and may separate into the form $\sum l(x_i)=0$. Detailed properties and examples are considered.
J. W. Tukey (Princeton, N. J.).

Mo, Ou Sing. Sur les moyennes hémisphériques. C. R. Acad. Sci. Paris **224**, 989–990 (1947).

If $M_t(U)$ denotes the mean value of U on the hemisphere $x=\{t^2-(y-y_0)^2-(z-z_0)^2\}^{\frac{1}{2}}$, then for the existence of a solu-

tion of the equation $tM_t(U)=W(y_0, z_0, t)$, it is necessary (1) that W admits the operations

$$y_0W+\partial\left\{\int_0^t t'Wdt'\right\}/\partial y_0, \quad z_0W+\partial\left\{\int_0^t t'Wdt'\right\}/\partial z_0$$

any number of times and in any order, and (2) that these two operations permute. This result is due to Hadamard [see Le Problème de Cauchy, Paris, 1932]. In the present note it is shown that (2) is a consequence of (1).

J. W. Calkin (Houston, Tex.).

***Kjellberg, Bo. On some inequalities.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 333–340. Jul. Gjellerups Forlag, Copenhagen, 1947.

It is well known that Holder's inequality implies the convexity in (α, β) of $I(\alpha, \beta)=\int|f(x)|^\alpha|g(x)|^\beta dx$. In the case in which $g(x)=x$ and the limits of integration are 0 and ∞, the author deduces that the convergence of $I(\alpha, \beta)$ at (α_1, β_1) and at (α_2, β_2) implies its convergence when (α, β) lies in a triangle which he terms the "shadow of convergence."

L. C. Young (Cape Town).

Obrechkoff, Nikola. Sur quelques inégalités pour les différences des fonctions d'une variable réelle. C. R. Acad. Sci. Paris **224**, 880–882 (1947). satisfying $\varphi^{(n)}(x)\leqq\psi^{(n)}(x)$

Let $\varphi(x)$ and $\psi(x)$ have nth derivatives for $x<a$ and let, for some sequence x_k, $x_k\rightarrow-\infty$, $\lim x_k^{-m}\varphi(x_k)=B$, $\lim x_k^{-m}\psi(x_k)=C$, where $0\leqq m<n$. Then

$$\varphi^{(m)}(x)-m!B\leqq\psi^{(m)}(x)-m!C$$

for $x<a$. This is obtained from a more general result in terms of differences and generalizes the author's earlier result [same C. R. **222**, 531–533 (1946); these Rev. **7**, 419].

R. P. Boas, Jr. (Providence, R. I.).

Obrechkoff, Nikola. Sur les solutions bornées de quelques équations intégrales singulières. C. R. Acad. Sci. Paris **224**, 993–995 (1947).

Let $p(x)\epsilon L(0, \infty)$, $p_1(x)=p(x)$, $p_n(x)=\int_0^x p_{n-1}(t)p(x-t)dt$, and suppose that, for each positive x,

$$\lim_{n\to\infty}\int_0^\infty|p_n(t)|dt=0, \quad \lim_{n\to\infty}\int_x^\infty|p_n(t)-p_n(t-x)|dt=0.$$

The author states the following theorems. (1) If, for $x>\alpha$, $f(x)=\int_0^\infty p(t)f(x+t)dt$ and $f(x)$ is bounded, $f(x)$ is a constant. (2) If $\varphi_n(x)=\int_0^\infty p(t)\varphi_{n+1}(x+t)dt$ and the $\varphi_n(x)$ are uniformly bounded for $x>\alpha$, then all the $\varphi_n(x)$ are equal to the same constant. The second theorem contains a result of Tagamlitzki [same C. R. **223**, 940–942 (1946); these Rev. **8**, 259]. Analogous results are stated for $(-\infty, \infty)$, for functions of several variables and for sequences.

R. P. Boas, Jr. (Providence, R. I.).

***Bang, Thøger. On quasi-analytic functions.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 249–254. Jul. Gjellerups Forlag, Copenhagen, 1947.

An account of some of the results of the author's thesis [Copenhagen, 1946; these Rev. **8**, 199]. In particular, the author describes his method for summing the Taylor series of a quasi-analytic function of class $C\{m_n\}$ on $(-\infty, \infty)$, namely

$$f(x)=\lim_{N\to\infty}\sum_{n=0}^{N-1}f^{(n)}(a)\frac{(x-a)^n}{n!}c_{N,n},$$

$$c_{N,n}=\int_{s_n}^{s_N}d\xi_n\int_{s_{n-1}}^{\xi_n}d\xi_{n-1}\cdots\int_{s_1}^{\xi_2}d\xi_1 \div\int_0^{s_N}d\xi_n\int_0^{\xi_n}d\xi_{n-1}\cdots\int_0^{\xi_2}d\xi_1,$$

$$s_n=\sum_{j=1}^n m_{j-1}/m_j.$$

R. P. Boas, Jr. (Providence, R. I.).

Mandelbrojt, Szolem. Sur les fonctions indéfiniment dérivables sur une demi-droite. C. R. Acad. Sci. Paris **224**, 1092–1093 (1947).

The author announces the following result. Let $\{\nu_n\}$ be a sequence of nonnegative integers with $\nu_1=0$, the complementary sequence having upper density $D<\frac{1}{2}$, and let $f(x)$ be of class C^∞ and be bounded, with its derivatives, on $0\leqq x<\infty$. Let $C(\sigma)=\sup_{n\geqq1}(n\sigma-\log m_n)$, $m_n=\sup_{0\leqq x<\infty}|f^{(n)}(x)|$. Then if $f^{(\nu_n)}(0)=0$, $f(x)\equiv0$ if

$$\int^\infty C(\sigma)\exp\left\{-\int^\sigma[1-2DC(u)]du\right\}d\sigma=\infty.$$

A still more general result is also stated, together with a theorem in the opposite direction showing that the result just stated cannot be sharpened much further.

R. P. Boas, Jr. (Providence, R. I.).

Theory of Sets, Theory of Functions of Real Variables

Eyraud, Henri. Le théorème du continu. C. R. Acad. Sci. Paris **224**, 85–87 (1947).

The author asserts a proof of the hypothesis of the continuum, namely $2^{\aleph_0}=\aleph_1$. One point in the sketch of the proof [line 8 from bottom of p. 86] was not clear to the reviewer. Final judgment must await the publication of full details.

J. W. Tukey (Princeton, N. J.).

Eyraud, Henri. Alignement des fonctions arithmétiques dans un réseau dénombrable bien ordonné. Ann. Univ. Lyon. Sect. A. (3) **9**, 27–31 (1946).

One arithmetic function f (i.e., from integers to integers) is said to majorise another function g if, for some n, $x\geqq n$ implies $f(x)>g(x)$. This relation is denoted by $f>g$. If $n=1$ we say that f majorises g totally and if for $x<n$ we have $f(x)\geqq g(x)$ we say that f majorises g semi-totally. Aligning f and g means replacing $f(x)$ by min $(f(x), g(x))$ or by max $(f(x), g(x))$. Let S be a series of arithmetic functions well-ordered by the relation $>$. Then S is necessarily at most countable if it has the following property: each f in S majorises (totally or semi-totally) all preceding functions. If S is countable we can modify the functions by alignments so as to obtain a new series of functions which has the property just mentioned. If there is a function h which is not majorised by any f in a countable S then we can construct, by alignments, a function H which majorises every f, attention being confined to an infinite subsequence of the integers.

J. Todd (London).

Eyraud, Henri. Divisibilité asymptotique des suites. Théorème de la croissance. Ann. Univ. Lyon. Sect. A. (3) **9**, 50–54 (1946).

Let A and B be two increasing sequences of integers; A is said to be a multiple of B (and B a divisor of A) if A

contains all but a finite number of the terms of B. Chains of divisors and multiples of a given sequence are studied. The following result is stated [cf. the preceding review for notation]. There exists a series (ordered by the relation $>$) of arithmetic functions of type Ω, such that an arbitrary arithmetic function is majorised by some term of the series.
J. Todd (London).

Borůvka, O. On decompositions of sets. Rozpravy II. Třídy České Akad. **53**, no. 23, 26 pp. (1943). (Czech)
Borůvka, O. Über Zerlegungen von Mengen. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **44** (1943), 330–343 (1944).

L'auteur appelle décomposition (Zerlegung) dans un ensemble la donnée de sous ensembles disjoints non assujettis à recouvrir tout l'ensemble. En ce qui concerne les décompositions sur un ensemble, c'est-à-dire les relations d'équivalence, l'auteur redémontre qu'elles forment un réseau (Verband) et que ce réseau est modulaire si on a la condition d'associabilité (Komplementär). Des relations d'équivalence forment une suite si chacune est compatible avec la précédente. Deux suites de relations d'équivalence sont dites complémentaires si chaque terme de l'une est associable avec chaque terme de l'autre. On peut appliquer la construction de Zassenhaus à deux telles suites. Elle donne des raffinements de même longueur réduite entre lesquels on peut définir une certaine correspondance biunivoque (verknüpft). Ces résultats ne différent pas essentiellement de ceux de P. Dubreil et M. L. Dubreil-Jacotin [J. Math. Pures Appl. (9) **18**, 63–95 (1939)]. *J. Kuntzmann* (Grenoble).

Borůvka, Otakar. Théorie des décompositions dans un ensemble. I. Publ. Fac. Sci. Univ. Masaryk **1946**, no. 278, 37 pp. (1946). (Czech. French summary)

Des sous ensembles disjoints de G, non assujettis à recouvrir complètement G, forment une décomposition dans G. L'auteur montre que si l'on a une famille de décompositions, on peut définir une union (plus petit recouvrement) et dans certains cas une intersection (plus grand raffinement). Un cas particulier important est celui où l'inclusion définit une correspondance biunivoque entre les sous ensembles formant le plus grand raffinement et les sous ensembles de chacune des décompositions. L'auteur étudie enfin le passage d'une décomposition dans G à une décomposition dans $G\times G$. *J. Kuntzmann* (Grenoble).

Climescu, Al. Sur les espaces à topologie transitive d'ordre n. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] **1**, 259–269 (1946).

A. Appert [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **23**, 135–142 (1937)] a appelé "espaces à topologie transitive" les espaces où la fermeture $\bar{A}=fA$ de tout sous-ensemble A est assujettie aux quatre axiomes suivants: $\bar{0}=0$ (0 désignant l'ensemble vide); $A\subset\bar{A}$; $\bar{B}\subset\overline{A+B}$; $\bar{\bar{A}}=\bar{A}$. L'auteur appelle "espaces à topologie transitive d'ordre n" les espaces qui s'obtiennent à partir des précédents en y remplaçant l'axiome $\bar{\bar{A}}=\bar{A}$ par l'axiome plus faible suivant:
$$\underbrace{ff\cdots f}_{n+1\text{ fois}}A=\underbrace{ff\cdots f}_{n\text{ fois}}A.$$
L'auteur donne des applications et des exemples de tels espaces en algèbre abstraite. Il définit et étudie dans ses espaces les notions d'ensemble fermé d'ordre i, d'ensemble ouvert d'ordre i, d'intérieur d'ordre i, d'extérieur d'ordre i, de frontière d'ordre i (i entier positif), et donne diverses règles de calcul. *A. Appert* (Saumur).

Pétresco, Julien. Ensembles ordinaires des espaces topologiques. Disquisit. Math. Phys. **5**, 65–73 (1946).

An "ordinary" set B is one whose frontier coincides with the frontier of its interior. The author studies unions and intersections of such sets. Apparently because he supposes that the frontier of an arbitrary set A can have no interior, many of the fundamental relations stated do not obtain. For example, 1.1 and 1.5 fail if A is the set of negative real numbers plus the set of rational numbers. [Cf. R. Paintandre, C. R. Acad. Sci. Paris **223**, 121–123 (1946); these Rev. **8**, 46.] *R. Arens* (Los Angeles, Calif.).

Herzog, F., and Bissinger, B. H. A Cantor function constructed by continued fractions. Bull. Amer. Math. Soc. **53**, 104–115 (1947).

Let $\{C_n\}$ denote the proper fraction the continued fraction for which has partial quotients C_n (whose number may be finite). If $x=\{c_n+1\}$ with all $c_n>0$ the authors define $\varphi(x)=\{c_n\}$. This defines φ in a set E: outside E, φ is defined so as to preserve continuity and monotony. Irrational points of E are divided into three classes: (i) c_n bounded, (ii) c_n unbounded, $\sum 1/c_n$ divergent, (iii) $\sum 1/c_n$ convergent. Results given are: (i) $D\varphi=\infty$, (ii) $1<D^-=D^+<\infty$, (iii) $D_+=D_-=0$, $D^+=D^-=\infty$, in the usual notation for derivatives. *H. D. Ursell* (Leeds).

Faedo, Sandro. Su gli insiemi chiusi di misura nulla. Ann. Scuola Norm. Super. Pisa (2) **10**, 249–252 (1941).

If P is a linear perfect set, numbers $a_n>0$ exist such that P cannot be covered by a sequence of intervals I_n of respective lengths a_n. Hence if, for a closed linear set F, it is possible for any set a_n to cover F with intervals I_n of lengths a_n, then F must be enumerable. *H. D. Ursell* (Leeds).

Faedo, Sandro. Il principio di Zermelo per lo spazio delle funzioni continue. Ann. Scuola Norm. Super. Pisa (2) **10**, 209–214 (1941).

The author sets down a rule for selecting a definite function from an arbitrary closed class of continuous functions of a single real variable. The functions of the class need not all be defined on the same interval, and the intervals of definition may be finite or infinite. *L. M. Graves.*

★**Faedo, Sandro. Il principio di Zermelo nello spazio hilbertiano.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 210–220. Edizioni Cremonense, Rome, 1942.

Cf. the preceding review.

Pettineo, B. Sull'esistenza di funzioni di accumulazione di un insieme di funzioni continue. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 301–306 (1946).

Let F_n ($n=1, 2, \cdots$) be a sequence of functions $f(x)$ continuous in the interval $[a, b]$, such that $F_n\supseteq F_{n+1}$. Generalizing the notion of equicontinuity, the author calls this sequence "similarly continuous" if to every sequence of positive numbers $\epsilon_n\to 0$ there exists a sequence of sets F_n^* ($n=1, 2, \cdots$) with $F_n^*\subseteq F_n$ and a sequence of positive numbers δ_n, such that the oscillation of every function of F_n^* is smaller than ϵ_n in every subinterval of $[a, b]$ whose length is smaller than δ_n. Then he proves that, in order that a set of functions $f(x)$ continuous and uniformly bounded in

$[a, b]$ possess there at least one "function of accumulation" (in the sense of uniform convergence), it is necessary and sufficient that there be an infinite sequence of sets of functions $f(x)$ which is similarly continuous. *A. Rosenthal.*

Pettineo, B. Sulle funzioni di accumulazione approssimate di un insieme di funzioni continue. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 916–918 (1946).

Let F be a set of real-valued functions f each defined on an interval I_f of the real line. A function φ is called a function of approximate accumulation of F if given ϵ and $\rho>0$ there is an infinite subset F' of F such that, for each f in F', I_f differs from I_φ by line segments of length less than ϵ and also $|f(x)-\varphi(x)|\leqq\rho$ for all x in $I_f\cap I_\varphi$ except those in some set of outer measure less than ϵ. This note gives a necessary and sufficient condition for existence of a function of approximate accumulation for a set F of continuous functions; the condition is in terms of a "quasi-equicontinuous sequence of subsets of F." *M. M. Day* (Urbana, Ill.).

Fáry, István. Un critère de compacité pour les fonctions continues. C. R. Acad. Sci. Paris 224, 992–993 (1947).

This paper considers the space of continuous functions $f(x)$ with the topology determined by ordinary convergence of sequences. This topology differs from that of weak convergence in the space $\mathfrak{C}$ in that norms of convergent sequences need not be bounded. The author gives the following necessary and sufficient conditions for compactness of a set E: (1) for each x the set of functional values $f(x)$ is bounded for f in E; (2) for each x the functions f in E are quasi-equicontinuous at x. The second condition means that for each $\epsilon>0$ and each sequence x_n approaching x, there exists N such that, for each f in E, $|f(x)-f(x_j)|<\epsilon$ for some j between 1 and N. There are a few obvious misprints. *L. M. Graves* (Chicago, Ill.).

Dell'Agnola, C. A. Sulla convergenza di una successione di aggregati. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 102, 425–442 (1943).

Incited by a paper of L. Amerio [Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 2, 699–702 (1941); these Rev. 8, 140] the author discusses the convergence of sequences of sets in the plane (with a generalization to n-dimensional Euclidean spaces). However, he was not aware that his results (with other notations) had already been obtained by F. Hausdorff [Mengenlehre, 2d ed., Berlin-Leipzig, 1927, pp. 145–150], even for more general spaces [cf. also P. Alexandroff and H. Hopf, Topologie I, Berlin, 1935, pp. 111–115]. In particular, Hausdorff had already proved the coincidence of metric and topological convergence in compact metric spaces. *A. Rosenthal* (Lafayette, Ind.).

Dell'Agnola, C. A. Le successioni di aggregati e il teorema fondamentale del calcolo integrale. Ist. Veneto Sci. Lett. Arti. Parte II. 104, 999–1030 (1946).

The author considers sequences $A_1, A_2, \cdots$, where A_n is an aggregate of numbers $a_{n1}, a_{n2}, \cdots, a_{nr_n}$. He defines convergence and establishes some elementary properties of such sequences. As an application the concept of the Riemann integral is discussed. *O. Szász* (Cincinnati, Ohio).

Dell'Agnola, C. A. Considerazioni sulle funzioni continue di una variabile. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 102, 727–748 (1943).

Let (1) $f_1(x), \cdots, f_n(x), \cdots$ be a sequence of functions continuous in $[a, b]$. Let h be fixed such that $0<h<b-a$ and let $\omega_n(x, h)$ be the oscillation of the function $f_n(x)$ in $[x, x+h]$; then $\omega_n(x, h)$ is defined and continuous for $x\epsilon[a, b-h]$. Moreover, let $\omega_n(h)$ be the maximum of $\omega_n(x, h)$ in $[a, b-h]$ and set $\limsup_{n\to\infty}\omega_n(h)=\bar{\omega}(h)$, $\liminf_{n\to\infty}\omega_n(h)=\underline{\omega}(h)$. Then the author proves that the functions (1) are equicontinuous (or quasi-equicontinuous) in $[a, b]$, if and only if $\lim_{h\to0}\bar{\omega}(h)=0$ or $\lim_{h\to0}\underline{\omega}(h)=0$, respectively. Here the author calls the functions (1) quasi-equicontinuous in $[a, b]$ if to every $\epsilon>0$ there is a $\delta>0$ such that for every positive $h<\delta$ infinitely many $\omega_{n_i}(x, h)$ $(i=1, 2, \cdots)$ are smaller than ϵ in $[a, b-h]$. *A. Rosenthal* (Lafayette, Ind.).

Fichera, Gaetano. Sull'integrazione delle funzioni. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 2, 336–347 (1941).

Starting from the definition of the integral given by M. Picone [Lezioni di Analisi Infinitesimale, Catania, 1923] the author generalizes that definition for quite arbitrary domains M of integration and for unbounded functions. He first defines the upper and lower integrals on M and then, in case of their coincidence, arrives at the integral. *A. Rosenthal* (Lafayette, Ind.).

Zorn, M. A. Approximating sums. Amer. Math. Monthly 54, 148–151 (1947).

Let g be a complex-valued function such that $g(t)$ is defined for $0\leqq t\leqq1$ and let C be the set of values of g. A single-valued function f is said to be an admissible integrand if f is differentiable on some open set containing C. With $0=t_0\leqq t_1\leqq\cdots\leqq t_n=1$ let $|\Delta;t|=\max(t_i-t_{i-1})$, $z_i=g(t_i)$, $\Delta z_i=z_i-z_{i-1}$, $|\Delta;z|=\max|\Delta z_i|$, $\|\Delta;z\|=\sum|\Delta z_i|^2$, $z_i'=g(t_i')$ where $t_{i-1}\leqq t_i'\leqq t_i$, and call $\sum f(z_i')\Delta z_i$ an approximating sum. The author summarizes his results in the following theorem. A necessary and sufficient condition for the convergence of approximating sums belonging to all admissible integrands is that $\lim\|\Delta;z\|$ is equal to zero. It is pointed out that g defined by $g(0)=0$ and $g(t)=t+it\sin(1/t)$ for $t\neq0$ is not of bounded variation but $\lim\|\Delta;z\|=0$. Problems for further investigation are given. *J. F. Randolph* (Oberlin, Ohio).

Nikodym, O. Remarques sur les intégrales de Stieltjes en connexion avec celles de MM. Radon et Fréchet. Ann. Soc. Polon. Math. 18, 12–24 (1945).

A necessary and sufficient condition on the function $\rho(x)$ of a real variable in order that the set function μ, defined for intervals of the form $E=(a, b]$ by $\mu(E)=\rho(b)-\rho(a)$, possess a countably additive extension to all Borel sets is that ρ be continuous on the right and of bounded variation. For real-valued functions ρ (and consequently for functions whose values are in a finite-dimensional real Banach space) this theorem is well known. The author proves the result for functions whose values are in a real Banach space R which satisfies the condition (*) whenever $x_n\epsilon R$, $n=1, 2, \cdots$ and for every subsequence $\{x_{k(n)}\}$ the series $\sum_{n=1}^\infty x_{k(n)}$ is convergent in R then $\sum_{n=1}^\infty\|x_n\|<\infty$. He points out that infinite-dimensional Hilbert space does not satisfy (*); it is easy to show similarly that none of the familiar infinite-dimensional Banach spaces (such as L_p, C, and M) satisfies (*). The reviewer conjectures that if R satisfies (*) then R is finite-dimensional. At the basis of this conjecture lies the closely related work of M. E. Munroe [Duke Math. J. 13, 351–365 (1946); these Rev. 8, 387] on the relation between unconditional and absolute convergence. *P. R. Halmos.*

Ridder, J. Über Stieltjessche Integrale und ihre Anwendung zur Darstellung linearer Funktionale. I. Nieuw Arch. Wiskunde (2) 22, 171–188 (1946).

Four definitions of Stieltjes integral type are given of which the first is due to Dushnik [On the Stieltjes Integral, Ann Arbor, Mich., 1931, pp. 13–14] and the third to the reviewer [Trans. Amer. Math. Soc. 36, 868–875 (1934), in particular, pp. 873–874]. For the second definition, in the approximating sums $\sum_{i=0}^{n-1} f(\xi_i)(\alpha(x_i')-\alpha(x_{i+1}))$ it is assumed that at points of discontinuity α is double valued: $\alpha(x+0)$, $\alpha(x-0)$ and $x+0$, $x-0$ may appear among the points x_i; also f may be similarly double valued at the same points. The limit process is equivalent to the Moore-Smith limit by successive subdivisions. The fourth integral is of the form $\int_a^b f d\psi$, where ψ is an additive function of Jordan measurable subsets of (a, b), the limit being by successive partitions. It is shown that the most general linear functional on the space of Riemann integrable functions with $\|f\|$ the least upper bound of $|f|$ in (a, b) is expressible as an integral of the fourth type. *T. H. Hildebrandt.*

***Buch, Kai Rander. Remarques sur les mesures dans les espaces abstraits et sur la théorie de la probabilité.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 259–264. Jul. Gjellerups Forlag, Copenhagen, 1947.

A lecture stating the principal results and outlining the methods of the author's earlier papers on the set of values of a finite measure. [Cf. Danske Vid. Selsk. Math.-Fys. Medd. 21, no. 9 (1945); these Rev. 7, 279.]

P. R. Halmos (Chicago, Ill.).

de Finetti, Bruno. Come si enunciano i primi teoremi dell'analisi svincolandosi dall'ipotesi della derivabilità. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 4, 25–33 (1943).

The author generalizes the theorems of Rolle, Cauchy, Darboux, de l'Hospital, etc., to the case in which differentiability is not assumed. Let $\underline{f}'(x)$ and $\bar{f}'(x)$ be the lower and upper derivates of $f(x)$, respectively, and denote by $\tilde{f}'(x)$ an arbitrary derivate of f at x. The author's generalization of Rolle's theorem is as follows. If $f(x)$ is continuous in $[a, b]$ and $f(a)=f(b)$, then there exists a $c\epsilon(a, b)$ such that $\underline{f}'(c)\leqq 0 \leqq \bar{f}'(c)$. He generalizes the theorem of de l'Hospital in the following way. Let $f(x)$ and $g(x)$ be defined in (a, b) and let both functions become either infinitesimal or infinite as $x\rightarrow b$; moreover, $g(x)$ shall not vanish and shall be monotone in (a, b); then

$$\underline{\lambda} \leqq \liminf_{x\rightarrow b} f(x)/g(x) \leqq \limsup_{x\rightarrow b} f(x)/g(x) \leqq \bar{\lambda},$$

where $\underline{\lambda}$ and $\bar{\lambda}$ are the lower and upper limits of the minimum and maximum, respectively, of $\tilde{f}'(x)/\tilde{g}'(x)$ as $x\rightarrow b$.

A. Rosenthal (Lafayette, Ind.).

Hruška, Václav. Une note sur les fonctions aux valeurs intermédiaires. Časopis Pěst. Mat. Fys. 71, 67–69 (1946). (French. Czech summary)

The author shows that, if $f'(x)$ exists and $\phi'(x)$ is strictly monotonic on a closed interval $[a, b]$, then

$$\lambda(x)=\{f'(x)-f'(a)\}/\{\phi'(x)-\phi'(a)\}$$

takes on every value between any two of its values on the interval. The proof is based on the Darboux intermediate value theorem. In an editorial remark, V. Jarník shows that $F'(x)/G'(x)$ has this intermediate value property under the weaker condition that $G'(x)\neq 0$ on the interval. The proof uses the mean value theorem; Darboux's result follows by putting $G(x)=x$. *L. W. Cohen* (Bayside, N. Y.).

Clarkson, J. A. A property of derivatives. Bull. Amer. Math. Soc. 53, 124–125 (1947).

If $f'(x)$ exists, real and finite, everywhere, then the set of x, $a<x<b$, $\alpha<f'(x)<\beta$, is always either empty or of positive Lebesgue measure. *A. J. Ward.*

Sélivanoff, N. A. Note sur les fonctions dérivées. Bull. [Izvestiya] Math. Mech. Inst. Univ. Tomsk 3, 125–127 (1946). (Russian and French)

The product of two derivatives does not have to be a derivative. However, the author shows that the product of a bounded derivative and a continuous function is a derivative. From this he deduces that the product of a derivative and a function having a bounded derivative is the derivative of a function. *František Wolf* (Berkeley, Calif.).

Corominas, Ernesto. On Peano's generalized derivatives. Revista Unión Mat. Argentina 12, 88–93 (1946). (Spanish)

General discussion and announcement of results on the generalized derivatives defined by

$$f(x+h)=f(x)+hf'(x)+\cdots+h^n f^{(n)}(x)/n!+o(h^n).$$

R. P. Boas, Jr. (Providence, R. I.).

Corominas Vigneaux, Ernest. Sur les coefficients différentiels d'ordre supérieur. C. R. Acad. Sci. Paris 224, 89–91 (1947).

The derivatives studied by A. Denjoy [Fund. Math. 25, 273–326 (1935)] are called "Peano derivatives" by the author. The function $f(x)$ is n-fold differentiable in the Peano sense at x if $f(x+h)=\sum_{k=0}^{n} h^k f^{(k)}(x)/k!+o(h^n)$. The author states that the classical mean value theorems and some other theorems can be generalized for the Peano derivatives and also for another system of derivation of a similar type. No proofs are indicated. *A. Rosenthal.*

Corominas Vigneaux, Ernest. Dérivation de Riemann-Schwarz. C. R. Acad. Sci. Paris 224, 176–177 (1947).

The author defines the nth derivative of Riemann-Schwarz of $f(x)$ as the limit of the nth difference of $f(x)$ if n of the $n+1$ points employed have distances from x which are infinitesimals of the same order and nonequivalent to each other. Under certain conditions the nth Riemann-Schwarz derivative coincides with the nth "Peano derivative" [see the preceding review]. No proofs are given.

A. Rosenthal (Lafayette, Ind.).

Banerjee, D. P. On a few generalisations of Weierstrass' non-differentiable functions. Bull. Calcutta Math. Soc. 38, 137–139 (1946).

The functions $f(x)=\sum_{r=0}^{\infty} a^r \cos^p (b^r \pi m x)$ and

$$g(x)=\sum_{r=1}^{\infty} a^r \cos^p \{1^q 3^q \cdots (2r-1)^q x\}/1^q 3^q \cdots (2r-1)^q$$

are shown to be continuous nondifferentiable functions under suitable conditions on the parameters. Similar examples, which are stated in terms of the Legendre polynomials, are offered; the proofs, however, are incomplete.

P. Civin (Eugene, Ore.).

Leontief, Wassily. A note on the interrelation of subsets of independent variables of a continuous function with continuous first derivatives. Bull. Amer. Math. Soc. **53**, 343–350 (1947).

Let f be a function of n variables denoted collectively by X. Let V' be a subset of X and V'' the complementary set. If functions ϕ, ψ exist such that, in some neighbourhood of a point, we have $f(x)=\psi\{\phi(V'), V''\}$, V' is said to be locally functionally separable (l.f.s.) there. A condition necessary and sufficient for this to be the case is that $(\partial/\partial x)[(\partial f/\partial y)/(\partial f/\partial z)]=0$ for all x in V'' and all y and z in V'. Suppose that X' and Y' are l.f.s. in X and that neither X' nor Y' is a subset of the other. The sets $A=X'-Y'$, $B=Y'-X'$, $C=X'Y'$ are then also l.f.s. Furthermore, if $X'+Y'$ is l.f.s. there exist functions α, β, γ such that $\phi(X'+Y')=\alpha(A)+\beta(B)+\gamma(C)$. All functions are assumed to have suitable differentiability properties and the proofs are made by partial differential calculus. The note arose from studies in mathematical economics. *J. Todd.*

Guareschi, Giacinto. Sul calcolo effettivo degli iperdifferenziali totali delle funzioni di più variabili reali. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **2**, 153–169 (1941).

Function $z=f(x_1, \cdots, x_n)$ is defined on a set I of $(x_1, \cdots, x_n)$ Euclidean space, determining locus I' in (x, z) space. The tangent space to I' at a point A' is the linear space of minimum dimension containing all rays through A' obtainable as limits of secants to I' through A' (corde improprie). (Thus the tangent space to a regular curve will be the usual tangent line.) As a tool in the study of tangent spaces, the total hyperdifferential is used [introduced by F. Severi, Ann. Mat. Pura Appl. (4) **13**, 1–35 (1934)]. If, corresponding to point A, there is a set of constants $\alpha_1, \cdots, \alpha_n$, such that, when M, N are any two points of I near A such that vector MN has components $\Delta x_1, \cdots, \Delta x_n$, ϵ is determined by $f(N)-f(M)=\alpha_1\Delta x_1+\cdots+\alpha_n\Delta x_n+\epsilon MN$, and as M, N approach A, $\lim \epsilon=0$, then function $f(x)$ is said to have total hyperdifferential $\alpha_1\Delta x_1+\cdots+\alpha_n\Delta x_n$ at A. For example, if I' is a smooth curve in 3-space, one of the constants α_1, α_2 can be taken arbitrarily, and the other is then uniquely determined. Severi [loc. cit.] has shown that a necessary and sufficient condition that f is hyperdifferentiable at A in I is that the tangent space to I' at A' (corresponding to A) does not contain a line parallel to the z-axis.

Corresponding to the hyperdifferential at $A(a_1, \cdots, a_n)$ is the $(n-1)$-plane (1) $z-f(A)=\alpha_1(x_1-a_1)+\cdots+\alpha_n(x_n-a_n)$. The tangent space at A' is the intersection of all the planes (1) determined by all possible hyperdifferentials at A. A tool in studying the hyperdifferential is the hyperdirectional derivative, also introduced by Severi [loc. cit.]. It is the limit, if it exists, of $\{f(N)-f(M)\}/MN$ as M, N approach A on I, in such a manner that the direction of the vector MN approaches as limit the direction in which the hyperdirectional derivative is being taken. Two examples are given. Analytic conditions for hyperdifferentiability are derived. They are quite long. *A. B. Brown.*

Guareschi, Giacinto. Sulle matrici funzionali formate con le iperderivate delle funzioni di più variabili reali e sull'inversione e sulla riduzione di un sistema di tali funzioni. Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **4**, 88–93 (1943).

Given the system (1) $x_i=f_i(u_1, \cdots, u_n)$, $i=1, \cdots, m$, where the f_i are defined on a set I of Euclidean u-space S_n, denote by I' the locus of (1) in the space of $u_1, \cdots, x_m$. If p variables $t_1, \cdots, t_p$ have as locus a set H in their space, they are said to be tangentially independent at a point Q of H if Q is a limit point of H and the tangent space to H at Q is the entire t-space. [For definitions, cf. the preceding review.] If the tangent space is of dimension $s<p$, then any s of the t's determining an s-space not perpendicular to the tangent space are tangentially independent and the remaining $p-s$ variables are tangentially dependent on the s of them thus chosen.

Let h be the dimension of the tangent space to I at a point A. Let A' correspond to A on I'. Suppose there is a subset $y_1, \cdots, y_{h+p}$ of the variables $u_1, \cdots, x_m$, such that, if $z_1, \cdots, z_{n+m-h-p}$ are the rest of those variables, then I' is given by (2) $z_q=\phi_q(y_1, \cdots, y_{h+p})$, $q=1, \cdots, n+m-h-p$, where (hypothesis) $y_1, \cdots, y_{h+p}$ are confined to a set $I^{(2)}$ of their space Σ_{h+p}, with $I^{(2)}$ in one-to-one correspondence with a neighborhood N of A' on I' by orthogonal projection from the points of N onto Σ_{h+p}. Then if the y's do not include any x_i, (2) is a reduced system of (1) relative to the (variables of the) space Σ_{h+p}. If the y's include one or more x_i, then (2) is called a system inverse to (1) with respect to $y_1, \cdots, y_{h+p}$ (totally inverse if the y's do not include any u's). Assuming that the f_i are totally hyperdifferentiable to the first order at A (as defined on I), and that the tangent space σ_h to I at A has dimension $h<n$, the author proves that the system (1) can be reduced to a system of $m+n-h$ functions tangentially dependent on h variables. Any h of the variables $u_1, \cdots, x_m$ determining a space not perpendicular to σ_h can be used as the variables of the latter set. Also, if $h\leqq n$, and if σ_h', the tangent space to I' at A', is not parallel to σ_h, then h variables can be found, tangentially independent as regards the projection of I' on their space, and including at least one x_i, with respect to which the system (1) can be inverted. Finally the author shows how, by use of a matrix involving hyperdirectional derivatives, one can choose h variables with respect to which (1) can be reduced or inverted. *A. B. Brown* (Flushing, N. Y.).

Colucci, Antonio. Sulla rappresentazione conforme delle superficie rettificabili. Rend. Accad. Sci. Fis. Mat. Napoli (4) **11**, 93–98 (1941).

Let S be a so-called rectifiable surface; that is, the coordinate functions (1) $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$ satisfy a Lipschitz condition in the domain D of definition. Rectifiable surfaces are quite amenable to analysis: such a surface S has a tangent plane at almost all points; the map on S of a rectifiable curve in D is a rectifiable curve; the area of S is given by the classical area integral. It is shown now that any rectifiable surface S admits a representation (1) which is conformal except at most on a set of superficial measure zero. *E. F. Beckenbach* (Los Angeles, Calif.).

Theory of Functions of Complex Variables

***Knopp, Konrad. Theory of Functions. II. Applications and Continuation of the General Theory.** Dover Publications, New York, 1947. x+150 pp. $1.50.

This is a translation by Frederick Bagemihl of the fourth German edition [Göschen, Berlin-Leipzig, 1931]. The foundations for the general theory of analytic functions were laid down in part I [cf. these Rev. **7**, 53]. In part II the author gives careful detailed discussions, including proofs and illustrations, of a few selected theorems, such as Weierstrass's factor theorem for entire functions, Mittag-Leffler's

partial fractions theorem for meromorphic functions and the theorem on the representation of elliptic functions in terms of the Weierstrass $\wp$-function. Similarly, the Riemann surfaces for $\log z$, $z^{1/n}$, $\sqrt{(z-a_1)\cdots(z-a_n)}$ are considered in detail, as well as certain properties of that for the general algebraic function. The author concludes by completing his discussion, begun in part I, of the general concept of analytic configuration. *L. H. Loomis* (Cambridge, Mass.).

Mayer, O. Sur le théorème fondamental de la théorie des fonctions de variable complexe. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi Iaşi] 1, 274–280 (1946).

A detailed proof of the strong form of Cauchy's integral theorem along lines, probably due originally to G. N. Watson [Complex Integration and Cauchy's Theorem, Cambridge University Press, 1914], of dividing the Jordan curve into a finite number of arcs, enclosing each in a polygon and using pieces of these polygons to obtain a Jordan polygon approximating the original curve from the inside.

L. H. Loomis (Cambridge, Mass.).

Reade, Maxwell O. On areolar monogenic functions. Bull. Amer. Math. Soc. 53, 98–103 (1947).

En se référant seulement à un travail récent [Haskell, Bull. Amer. Math. Soc. 52, 332–337 (1946); ces Rev. 7, 381], l'auteur appelle fonction aréolaire monogène une fonction $f(z)=u(x,y)+iv(x,y)$ dans le cercle-unité, avec dérivées secondes continues telles que $u''_{xy}=-\frac{1}{2}(v''_{xx}-v''_{yy})$. Affaiblissant la suffisance d'un critère de Haskell [loc. cit.], il démontre qu'une condition nécessaire et suffisante est que, quel que soit r $(0<r<1)$, $(\pi r^2)^{-1}\int_{C_r} f dz$ (où f est continue et C_r le cercle de centre S et rayon r) soit analytique dans le cercle $|\zeta|<1-r$. Puis il traite la même question en remplaçant C_r par un polygone régulier. [Il n'est pas mentionné le rapport avec la dérivée aréolaire de Pompeiu [Rend. Circ. Mat. Palermo 33, 108–113 (1912)] et que les fonctions monogènes aréolaires sont celles que Théodoresco appelle polynomes aréolaires du premier ordre; voir Ann. Roumaines Math. cahier 3, 3–62 (1936), p. 19.]

M. Brelot (Grenoble).

Pompeiu, D. De la définition du pôle en théorie des fonctions. Disquisit. Math. Phys. 1, 57–59 (1940).

Serghiesco, Stefan. Sur une intégrale curviligne donnant le nombre des zéros et des pôles distincts d'une fonction méromorphe dans un contour fermé. C. R. Acad. Sci. Paris 224, 440–442 (1947).

It is remarked that the integral

$$I=\int_C \frac{F'^2-FF''}{FF'}dz$$

counts the number of zeros and poles of a meromorphic function $F(z)$ enclosed by C with simple multiplicity. Unfortunately, there is a correction term coming from multiple values which are not zeros or poles. *L. Ahlfors.*

Vekua, N. On some mixed boundary problems of the theory of analytic functions. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 6, 245–254 (1945). (Georgian. Russian summary)

In the z-plane, let D be a simply-connected domain containing the origin and bounded by a simple closed curve C. Let C be divided into $2n$ nonoverlapping arcs by points $a_1, b_1, \cdots, a_n, b_n$ and let $L\equiv a_1b_1+\cdots+a_nb_n$, $L'\equiv b_1a_2+\cdots+b_na_1$. Let functions $f_1(t)$ and $f_2(t)$ be defined and satisfy Hölder conditions on L and L', respectively, and together satisfy Hölder conditions at end-points. The author gives a solution of the problem of determining a function $\varphi(z)=u+iv$, analytic in D, normalized by the condition $\varphi(0)=0$, remaining continuous on $D+L+L'$ except perhaps at end-points, where we have the local condition $|\varphi(z)|<C/|z-b_k|^\alpha$, $0\leqq\alpha<1$, and satisfying $u=f_1(t)$ on L' and $v=f_2(t)$ on L. *E. F. Beckenbach.*

Jabotinsky, Eri. Sur la représentation de la composition de fonctions par un produit de matrices. Application à l'itération de e^z et de e^z-1. C. R. Acad. Sci. Paris 224, 323–324 (1947).

With $F(z)=\sum_0^\infty a_nz^n$ is associated the matrix $\|\varphi_{m,n}\|$, where $\{F(z)\}^m=\sum_0^\infty \varphi_{m,n}z^n$. The iterates of $F(z)$ are then associated with powers of this matrix. [Cf. Bennett, Ann. of Math. (2) 17, 23–60 (1915).] The investigation of the powers may be facilitated by a matrix transformation. In particular, the matrix associated with $F(z)=a(e^z-1)$ can be expressed as a transform of a matrix involving positive integers, the "Stirling numbers." When $a=1$ this matrix may in turn be expressed with advantage as the sum of the unit and another matrix. Corresponding information follows for the expansions in powers of z of the iterates of $a(e^z-1)$. In particular, the pth iterate of e^z-1 has the expansion $\sum_1^\infty\sum_0^{n-1}\binom{p}{s}\omega_{s,n}z^n/n!$, where the $\omega_{s,n}$ are positive integers. Two expansions for the iterates of e^z are derived. *A. J. Macintyre* (Aberdeen).

Sherman, Seymour. A note on stability calculations and time lag. Quart. Appl. Math. 5, 92–97 (1947).

In the study of some control systems involving retarded action or time lags, it is sufficient for stability to ensure that the real equation $w(z)\equiv az^2+bz+\beta ze^{-z}+c=0$ shall have only roots with negative real parts. In the present paper it is shown that this is the case if $a>0$, $b\geqq|\beta|>0$ and $c>0$, or if $a>0$, $(bc\beta)\neq0$, $w(iy)\neq0$ for y real and

$$1-\operatorname{sgn}\{c(b+\beta)\}$$
$$+2\sum_{j=1}^{M}\operatorname{sgn}\{[-ay_j^2+c+\beta y_j\sin y_j][\beta\sin y_j]\}=0,$$

where the y_j are the roots of the equation $\Im w(iy)=0$. The proof is based on the Cauchy index theorem.

M. Marden (Milwaukee, Wis.).

Papaspyros, A. G. On doubly connected regions and the integral inequality of Ahlfors. Bull. Soc. Math. Grèce 21, 48–52 (1941). (Greek)

Let R be a doubly-connected region in the z-plane, separating 0 from ∞, and let α be its module. Map R into a curvilinear strip S by $Z=\log z$ and let $\Theta(X)$ be the length of the segment on $\Re(Z)=X$ in S as in Ahlfors' distortion theorem. Using this theorem, the author obtains the inequality

$$\int_{X_1}^{X_1+2\pi}\frac{dx}{\Theta(x)}\leqq 2\pi/\alpha.$$

R. P. Boas, Jr., and M. Heins (Providence, R. I.).

Kufareff, P. P. Zur Frage nach dem Verhalten der abbildenden Funktion am Rande. Bull. [Izvestiya] Math. Mech. Inst. Univ. Tomsk 3, 37–60 (1946). (Russian. German summary)

The author states as his chief result the following. Let G be a simply connected bounded region of the z-plane, whose boundary Γ satisfies the following conditions: (a) Γ is a closed Jordan curve; (b) Γ lies in the ring $1-\epsilon/2\leqq|z|\leqq1+\epsilon/2$;

(c) on the arc Γ_{φ_0} of the boundary Γ which lies in the angle $-\varphi_0 \leqq \varphi \leqq \varphi_0$ the radius vector to the point is a single-valued function of the argument, $r=f(\varphi)$, satisfying the condition $|f(\varphi)-1| \leqq Q|\varphi|^{\epsilon}$. There are assumed to be certain restrictions on the constants ϵ, φ_0, Q. Then a function $w=\Phi(z)$, $\Phi(0)=0$, mapping the region G on the unit circle $|z|<1$ and its derivatives $\Phi'(z), \cdots, \Phi^{(n-2)}(z)$ approach definite limiting values $\Phi(1), \Phi'(1), \cdots, \Phi^{(n-2)}(1)$ as z approaches 1 from within G along paths not tangent to Γ. Inequalities on the constants $\Phi(1), \Phi'(1), \cdots, \Phi^{(n-2)}(1)$ are also given. Since some of the proofs and preliminary theorems are incorrect, the validity of the results is open to question.

W. Seidel (Rochester, N. Y.).

Nehari, Zeev. The elliptic modular function and a class of analytic functions first considered by Hurwitz. Amer. J. Math. 69, 70–86 (1947).

The object of this paper is to obtain properties of the elliptic function $J(z)$ defined by means of the functional equation $J(z)=4\sqrt{J(z^2)}/\{1+\sqrt{J(z^2)}\}^2$. The author derives from these properties some general theorems concerning the functions which are regular in the unit circle $|z|<1$ and which do not vanish except for $z=0$. *P. J. Myrberg.*

Schilling, O. F. G. Ideal theory on open Riemann surfaces. Bull. Amer. Math. Soc. 52, 945–963 (1946).

Let Σ be an open Riemann surface. Denote by $F(\Sigma)$ the field of meromorphic functions on Σ and by $O(\Sigma)$ the subring of $F(\Sigma)$ consisting of the everywhere holomorphic functions. A topology is defined on $F(\Sigma)$ by means of the notion of uniform convergence on every compact subset of Σ. The author considers the ideal theory of the ring $O(\Sigma)$ in certain special cases: (a) $\Sigma=\Sigma_0$ is the complex plane; (b) Σ is the Riemann surface of a finite algebraic extension K of $F(\Sigma_0)$; (c) Σ results from a closed Riemann surface by omitting one point (or a finite number of points). In all these cases, it turns out that the closed ideals are the ideals determined by assigning certain sets of zeros and poles (with multiplicities); i.e., if A is a closed ideal, then A consists of all functions which at every place p of the surface have an order at least equal to some integer $v(p)$ given in advance; in this way a one-to-one correspondence is established between the closed ideals and certain groups of valuation vectors. Moreover, it is proved that, in cases (a) and (b), every prime ideal in $O(\Sigma)$ (closed or not) is maximal, and, in case (c), that every closed ideal is principal.

C. Chevalley (Princeton, N. J.).

***Lokki, Olli. Über eine Klasse von analytischen Funktionen.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 213–224. Jul. Gjellerups Forlag, Copenhagen, 1947.

Let E_n be the class of functions $f(z)$ regular in $|z|<1$ and such that $f(z_\nu)=w_\nu$, $\nu=1, \cdots, n$; $|z_\nu|<1$, and

$$I(f)=\pi^{-1}\int_0^1\int_0^{2\pi}|f(re^{i\theta})|^2 r\,dr\,d\theta<\infty.$$

(The number n may be ∞.) If n is finite, the minimum I_n of $I(f)$ is attained for a unique rational function of the form $\sum_{\nu=1}^{n} a_\nu(1-\bar{z}_\nu z)^{-2}$. Let z_0 be a number with $|z_0|<1$ different from all the z_ν ($\nu=1, \cdots, n$) and put $w_0=u_0+iv_0=f(z_0)$. If $(u_0, v_0, I(f))$ are interpreted as coordinates of a point P in a Cartesian coordinate system, then the points P fill a closed paraboloid of revolution as $f(z)$ varies through all functions of E_n. To a point on the boundary of this paraboloid corresponds a uniquely determined function $f(z)$; to a point inside correspond infinitely many functions. There are functions of class E_∞ if and only if $I_\infty=\lim_{n\to\infty} I_n<\infty$. If, in addition, $\sum(1-|z_\nu|)=\infty$, E_∞ contains only one function. If $\sum(1-|z_\nu|)$ converges, E_∞ contains infinitely many functions and there are close analogues of the theorems stated above for E_n. The method used is related to that used by Kakeya [Proc. Phys. Math. Soc. Japan (3) 3, 48–58 (1921)]. See also F. Riesz [Acta Math. 42, 145–171 (1919)].

W. H. J. Fuchs (Swansea).

Pisot, Charles. Propriétés arithmétiques des coefficients des séries de Taylor. C. R. Acad. Sci. Paris 224, 438–440 (1947).

Let $\sum_0^\infty a_n z^n$ be the expansion around the origin of a function $f(z)$, meromorphic for $|z| \leqq R$, where $R>1$. It is shown that, if the coefficients a_n take only a finite number of values (mod 1), $f(z)$ is rational. This statement includes Borel's theorem, a particular case of the Carlson-Pólya theorem (in which the a_n are rational integers), and a particular case of Szegö's theorem (in which the a_n take a finite number of values). *R. Salem* (Cambridge, Mass.).

***Whittaker, J. M. Series of Polynomials.** Edited by M. Mursi. Fouad I University, Faculty of Science, Cairo, 1943. iii+43 pp.

This brochure is the result of a series of lectures given by the author at the Fouad I University. It is closely linked to an earlier work of his [Interpolatory Function Theory, Cambridge University Press, 1935] and can be regarded essentially as bringing up to date chapter I of that work. The sequence of polynomials $\{p_n(z)\}$ ($n=0, 1, \cdots$) is a basic set if every polynomial is uniquely representable as a finite linear combination of the $p_n(z)$'s. Let (1) $z^n=\sum_{j=0}^{\infty}\pi_{nj}p_j(z)$ ($n=0, 1, \cdots$) (each series contains only a finite number of terms). If $f(z)=\sum_0^\infty \alpha_n z^n$ is analytic about $z=0$, then formally

$$f(z)=\sum_{n=0}^{\infty}[\Pi_n f(0)]p_n(z), \tag{2}$$

$$\Pi_n f(0)=\sum_{j=0}^{\infty}\pi_{jn}f^{(j)}(0)/j!. \tag{3}$$

Series (2) is called the basic series for $f(z)$ associated with the basic set $\{p_n(z)\}$. [All this as in chap. I of "Interpolatory Function Theory."] The present brochure is devoted to the study of problems on the convergence of basic series.

For each n let N_n be the number of nonzero coefficients π_{nj} in (1). The condition (4) $N_n^{1/n}\to 1$ is called Cannon's condition and the corresponding basic series (2) is then termed Cannon's series. The assumption of Cannon's condition has a unifying effect and permits the establishment of the following theorem. If (4) holds, then in order that the basic series (2) converge uniformly to $f(z)$ in $|z| \leqq R$ for every function $f(z)$ that is regular in $|z| \leqq R$, it is necessary and sufficient that $\lambda(R)=R$. Here $\lambda(R)=\limsup_{n\to\infty}\{\omega_n(R)\}^{1/n}$, $\omega_n(R)=\sum_{j=0}^{\infty}\pi_{nj}M_j(R)$, $M_j(R)=\max|p_j(z)|$ on $|z|=R$.

Another result: if (4) holds, then the following analogue of the Hadamard three-circle theorem is satisfied:

$$\lambda(R^{1+a}) \leqq [\lambda(R)]^{1-a/b}[\lambda(R^{1+b})]^{a/b}. \tag{5}$$

There are many other results, with and without the Cannon assumption. Some deal with convergence in the neighborhood of $z=0$, others with convergence in (or in and on the boundary of) a fixed circle centered at $z=0$; some use one or other of the conditions (6) $D_n/n\to 1$, $D_n=O(n)$, where D_n is the maximum degree of those polynomials $p_j(z)$ present (with nonzero multipliers π_{nj}) in the representation (1) for

the function z^n. (Each of (6) implies the Cannon condition (4).) Uniqueness is also considered. Results are also obtained for more general basic series than those considered above, by using the functions $F_n(R)$, $K(R)$ in place of $\omega_n(R)$, $\lambda(R)$. Here

$$F_n(R)=\max_{i,j}\max_{|z|=R}\left|\sum_{s=i}^{j}\pi_{n,s}p_s(z)\right|,\quad K(R)=\limsup_{n\to\infty}\{F_n(R)\}^{1/n}.$$

There are a number of misprints and some statements lacking in clarity (for example, the definition of D_n on page 13), but the reader can take these in his stride.

I. M. Sheffer (State College, Pa.).

Pollard, Harry. Representation of an analytic function by a Laguerre series. Ann. of Math. (2) **48**, 358–365 (1947).

The author solves the problem of finding necessary and sufficient conditions that the formal Laguerre series of an analytic function $f(z)$, $f(z)\sim\sum a_nL_n(z^2)$, converges to $f(z)$ in the strip $|y|<\tau$, $\tau>0$. The conditions proved are formulated as follows: it is necessary and sufficient that (a) $f(z)$ is analytic and even in the strip and (b) to every β, $0\leqq\beta<\tau$, there corresponds a positive number $B(\beta)$ such that, for $|y|\leqq\beta, -\infty<x<\infty, |f(z)|\leqq B(\beta)\exp\{x^2/2-|x|(\beta^2-y^2)^{\frac{1}{2}}\}$.

E. Kogbetliantz (New York, N. Y.).

Kober, H. Approximation of continuous functions by integral functions of finite order. Trans. Amer. Math. Soc. **61**, 293–306 (1947).

A function $f(x)$ is said to be uniformly bounded of order n if it is bounded in some interval and $\Delta_h{}^nf(x)$ is bounded in $-\infty<x<\infty$, uniformly for h in some interval; $f(x)$ is uniformly continuous of order n if in addition it is measurable and $\Delta_h{}^nf(x)\to0$ as $h\to0$, uniformly in $-\infty<x<\infty$. The main theorem of the paper is that $f(x)$, uniformly bounded of order n, can be approximated by entire functions $g_\alpha(x)$ of exponential type α (in the sense that $\sup_{-\infty<x<\infty}|f(x)-(x+i)^mg_\alpha(x)|\to0$ as $\alpha\to\infty$) if and only if $f(x)$ is uniformly continuous of order n. Here m is any nonnegative integer. An application is made to approximation by entire functions to functions analytic in a half plane. The theorem implies that if $f(x)$ is uniformly continuous of order n and uniformly bounded of order m $(m<n)$ then $f(x)$ is uniformly continuous of order m if $m\geqq1$, of order 1 if $m=0$. The author remarks that it would be interesting to have an elementary proof of this fact.

R. P. Boas, Jr. (Providence, R. I.).

Shahinian, A. L. On one test of incompletness of the analytic functions. Acad. Sci. Armenian SSR. Proc. [Doklady] **5**, 97–100 (1946). (Russian. Armenian and English summaries)

The author considers a region D topologically equivalent to the region between two circles which are tangent internally. Let D_2 be the class of functions $f(z)$ analytic in D and such that $\iint_D|f(z)|^2dxdy<\infty$, D^* the subclass satisfying a certain condition restricting their rate of growth in the neighborhood of the point of contact of the curves bounding D. The author gives conditions on the region which prevent D^* from being complete in D. *R. P. Boas, Jr.*

***Ríos, Sixto. Theory of the analytic continuation of Dirichlet series.** Centro Estudos Mat. Fac. Ci. Pôrto. Publ. no. **10**, 115 pp. (1947). (Portuguese)

A Portuguese edition of the author's lectures [cf. Revista Acad. Ci. Madrid **37** (1943); The Analytic Continuation of the Dirichlet-Stieltjes Integral, Madrid, 1944; these Rev. **7**, 61, 294]. *R. P. Boas, Jr.* (Providence, R. I.).

***Bejar Alamo, Juan. Funciones Definidas por Series de Dirichlet con Exponentes Complejos. [Functions Defined by Dirichlet Series with Complex Exponents].** Memorias de Matemática del Instituto "Jorge Juan," no. 2. Madrid, 1946. 125 pp.

This thesis is concerned with series $\sum a_ne^{-\lambda_ns}$ where the λ_n are not necessarily real. Classical theorems are extended under various hypotheses. Regions of convergence and overconvergence are discussed, particularly under hypotheses introduced by Hille [Ann. of Math. (2) **25**, 261–278 (1924)] and Schnee [Über irreguläre Potenzreihen und Dirichletsche Reihen, Inauguraldissertation, Berlin, 1908]. The location of singularities is discussed under the hypothesis that $\lambda_n=\alpha_n+i\beta_n$ with $\beta_n/\alpha_n\to0$, $\alpha_n\uparrow\infty$, $n/\lambda_n\to D$; in this connection the entire function $\prod_{n=1}^{\infty}(1-z^2/\lambda_n{}^2)$ is studied.

R. P. Boas, Jr. (Providence, R. I.).

***Iglesias Garrido, Tomas. Estudio de la Reordenación de Series de Dirichlet. [Study of the Rearrangement of Dirichlet Series].** Memorias de Matemática del Instituto "Jorge Juan," no. 1. Madrid, 1946. 111 pp.

This thesis is concerned with series $\sum a_ne^{-\lambda_ns}$ where the (real) sequence $\{\lambda_n\}$ is not necessarily monotonic. Classical theorems are extended under various hypotheses on $\{\lambda_n\}$. The principal topics are the regions of convergence, determination of singular points from the coefficients, overconvergence, analytic continuation by rearrangement. Since the series considered can be thought of as rearrangements of Dirichlet series, all the results can be interpreted as results about Dirichlet series in the usual sense.

R. P. Boas, Jr. (Providence, R. I.).

Bejar Alamo, Juan. On Dirichlet series with complex exponents. Revista Mat. Hisp.-Amer. (4) **7**, 70–86 (1947). (Spanish)

Iglesias, Tomas. On the rearrangement of Dirichlet series. Revista Mat. Hisp.-Amer. (4) **7**, 21–40 (1947). (Spanish)

Summaries of the authors' theses reviewed above.

Hadwiger, H. Ueber einen funktionentheoretischen Umordnungssatz von S. Ríos. Revista Mat. Hisp.-Amer. (4) **6**, 235–239 (1946).

A lecture devoted to Ríos's construction of a series of polynomials which can be rearranged to converge to any given analytic function in any region [see Revista Mat. Hisp.-Amer. (4) **3**, 100–128 (1943); Abh. Math. Sem. Hansischen Univ. **15**, 57–81 (1943); these Rev. **5**, 66; **6**, 268].

R. P. Boas, Jr. (Providence, R. I.).

Călugăreanu, Georges. Sur le problème des singularités des fonctions analytiques. Disquisit. Math. Phys. **4**, 95–104 (1945).

Let $f(z)$ be regular outside the lemniscate

$$|P(z)|=|z^n+\cdots|=\rho.$$

Expansions of the type $f(z)=\sum_{\nu=1}^{\infty}Q_\nu(z)P(z)^{-\nu}$ are studied, where $Q_\nu(z)$ is a polynomial of degree $n-1$. The relation of these expansions to the Chebyshev polynomials associated with the closed set S of the singular points of $f(z)$ is studied.

G. Szegö (Stanford University, Calif.).

Comét, Stig. Une propriété algébrique des équations de Cauchy-Riemann. C. R. Acad. Sci. Paris **224**, 623–625 (1947).

Relations of the form $dy=\theta dx$ are studied, where $y=(y_1, \cdots, y_n)$ is a function of the n real variables $x_1, \cdots, x_n$ and where θ is the matrix $(\partial y_i/\partial x_k)$. For $n=2$ the Cauchy-Riemann equations are obtained if $\theta=E\theta E'$, where

$$E=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

and E' is the transpose of E. A generalisation of the Cauchy-Riemann equations is obtained for arbitrary n by considering matrices which satisfy a relation $\sum \epsilon_i E_i \theta E_i'=0$, where the E_i are $n\times n$ matrices which form a representation of the base of a hypercomplex system. The system of equations satisfied by the "regular" functions of a quaternion variable are of this type. *O. Todd-Taussky* (London).

Hua, L. K. On the extended space of several complex variables. I. The space of complex spheres. Quart. J. Math., Oxford Ser. **17**, 214–222 (1946).

If we take the ordinary space of complex variables $z_1, \cdots, z_n$ there are different methods of completing it to a compact complex manifold by adding "points at infinity." In one case, every homeomorphism of the completed space has necessarily the form $z_i'=(\alpha_i z_{\theta}+\beta_k)/(\gamma_k z_k+\delta_k)$, $1\leqq i\leqq n$, $1\leqq k\leqq n$, and in another it must have the projective form

$$z_i'=(c_1^i z_1+\cdots+c_n^i z_n+c_0^i)/(c_1^0 z_1+\cdots+c_n^0 z_n+c_n^0).$$

The author proposes to study other, less obvious, cases showing analogous properties; in the present paper he analyzes the complex "Lie geometry of hyperspheres." Take $n+2$ complex variables $x_1, \cdots, x_n, y_1, y_2$; all linear homogenous transformations leaving the quadratic relation $\sum_1^n x_i^2 - y_1 y_2=0$ invariant; and then reduce the complex dimension by one by considering the quotients of the variables. This leads to a completion at infinity and the author shows that on the completed space the stated transformations are the only homeomorphisms present.

S. Bochner (Princeton, N. J.).

Theory of Series

Agnew, Ralph Palmer. Subseries of series which are not absolutely convergent. Bull. Amer. Math. Soc. **53**, 118–120 (1947).

Suppose that the series $\sum_1^\infty a_k$, where the a_k are complex, does not converge absolutely. It is shown that there exists an increasing sequence of integers n_k, with $n_{k+1}-n_k\to\infty$, such that $\sum a_{n_k}$ diverges. *W. W. Rogosinski.*

Andreoli, Giulio. Sulla convergenza delle serie ed i suoi caratteri gruppali (convergenza assoluta, incondizionata, totale). Rend. Accad. Sci. Fis. Mat. Napoli (4) **9**, 172–176 (1939).

A series $\sum a_n$ of complex numbers is "totally convergent" if each subseries is convergent, that is, if $\sum u_n a_n$ is convergent whenever $u_1, u_2, \cdots$ is a sequence of zeros and ones. A series of complex numbers is totally convergent if and only if it is absolutely (or unconditionally) convergent. It is observed that if $\sum a_n$ converges absolutely and u_n and v_n are bounded sequences, then $\sum u_n a_n$, $\sum v_n a_n$, $\sum (u_n+v_n)a_n$ and $\sum u_n v_n a_n$ are convergent; this is used as a basis for remarks about additive and multiplicative groups.

R. P. Agnew (Ithaca, N. Y.).

Barba, G. Proprietà gruppali nelle serie di Dirichlet, serie di Dirichlet gruppali. Rend. Accad. Sci. Fis. Mat. Napoli (4) **10**, 173–179 (1940).

The formal sum $\sum_{n=1}^\infty a_n e^{-\lambda_n s}+\sum_{n=1}^\infty b_n e^{-\lambda_n s}=\sum_{n=1}^\infty c_n e^{-\lambda_n s}$ of two Dirichlet series of the same type λ_n is always a Dirichlet series of the same type with $c_n=a_n+b_n$. The formal Dirichlet product

$$\sum_{j=1}^\infty a_j e^{-\lambda_j s}\sum_{k=1}^\infty b_k e^{-\lambda_k s}=\sum_{n=1}^\infty\Big\{\sum_{\lambda_j+\lambda_k=\mu_n} a_j b_k\Big\}e^{-\mu_n s}$$

of two series of type λ_n has type λ_n for every set of coefficients a_n and b_n if and only if $\lambda_j+\lambda_k$ is an element of the sequence λ_n whenever λ_j and λ_k are. These remarks lead the author to further remarks concerning sequences λ_n of complex numbers having the required property.

R. P. Agnew (Ithaca, N. Y.).

Schur, Issai. Identities in the theory of power series. Amer. J. Math. **69**, 14–26 (1947).

Let $g(x)=1+\sum_{n=1}^\infty a_n x^n$, $(g(x))^n=\sum_{\rho=0}^\infty a_{n\rho}x^\rho$, $n=0, \pm 1, \pm 2, \cdots$, $\phi_n(x)=\sum_{\nu=0}^\infty a_{n-\nu,\nu}x^\nu$, $f(x)=xg(x)$. Then, for every integral n, $f'(x)\phi_n(f(x))=(g(x))^{n+1}$. This follows from Cauchy's theorem [as was pointed out by G. Schur]. It furnishes for $y=f(x)=xg(x)$ the inversion $x=\sum_{\nu=1}^\infty \nu^{-1}a_{-\nu,\nu-1}y^\nu$. All these identities can be derived from Lagrange's inversion formula. The main purpose of the present paper is to furnish a purely arithmetic approach to Lagrange's formula and the consequences mentioned. This is based on the observation that the inverse matrices of $M=(a_{\mu,\mu-\nu})$, $N=(a_{\nu,\mu-\nu})$ can be represented in the form

$$M'=(\nu^{-1}\mu a_{-\nu,\mu-\nu}),\quad N'=(\mu^{-1}\nu a_{-\mu,\mu-\nu}),\quad \mu, \nu=1, 2, 3, \cdots.$$

The inversion of $w=zg(z^{-1})$ in the form

$$z=w+A_1+A_2w^{-1}+\cdots$$

is also obtained. Furthermore, a generalization of Lagrange's formula is given in which x^k is represented in terms of powers of y not only for positive integral but for arbitrary k. Finally these results are applied to the Faber polynomials to which a previous paper of the author [same J. **67**, 33–41 (1945); these Rev. **6**, 210] was devoted. *G. Szegö.*

Karamata, J. A note on convergence factors. J. London Math. Soc. **21**, 162–166 (1946).

Let $u_1\geqq u_2\geqq\cdots$, $u_n>0$, $s_1=0$,

$$s_{n+1}=\sum_{k=1}^n e_k u_k,\quad E_n=n^{-1}\sum_{k=1}^n e_k, \tag{1}$$

where the e_k are complex numbers. These formulas imply that

$$E_n=n^{-1}\sum_{k=1}^n (u_k^{-1}-u_{k+1}^{-1})s_{k+1}+n^{-1}s_{n+1}/u_{n+1}. \tag{2}$$

The problem is to find conditions which imply that $\lim E_n=0$. H. Rademacher [Math. Z. **11**, 276–288 (1921)] has shown that, if $s_n=s+o(1)$, $\lim u_n=0$ and $\lim\inf nu_n>0$, then $\lim E_n=0$. W. H. J. Fuchs [Proc. Edinburgh Math. Soc. (2) **7**, 27–30 (1942); these Rev. **4**, 79] has shown that, if $s_n=s+O(u_n)$, $\lim u_n=0$, $\sum u_n$ diverges and $\lim u_n/u_{n+1}=1$, then $\lim E_n=0$. Karamata's paper gives two more criteria. If $0\leqq\alpha<1$, $0\leqq\beta<1$, $\lambda=(1-\beta)/(1-\alpha)$, $s_n=s+o(n^\beta u_n^\alpha)$ and $\lim\inf n^\lambda u_n>0$, then $\lim E_n=0$. If $s_n=s+o(nu_{n-1})$ and $u_n=O(u_{2n})$, then $\lim E_n=0$. The proofs involve applications of the Silverman-Toeplitz-Schur theory of summability to the transformation (2). *R. P. Agnew* (Ithaca, N. Y.).

Vermes, P. Product of a T-matrix and a γ-matrix. J. London Math. Soc. 21, 129–134 (1946).

Let $A=(a_{nk})$ be a T-matrix, that is to say, a matrix that transforms every convergent sequence $\{s_k\}$ into a convergent sequence $\{t_n\}$ by means of the relations $t_n=\sum_{k=1}^{\infty}a_{nk}s_k$, $n=1, 2, 3, \cdots$, with $\lim s_k=\lim t_n$. Let $G=(g_{nk})$ be a γ-matrix, namely, a matrix that transforms every convergent series $\sum u_k$ into a convergent sequence $\{v_n\}$ by means of the relations $v_n=\sum_{k=1}^{\infty}g_{nk}u_k$, $n=1, 2, 3, \cdots$, with $\lim v_n=\sum u_k$. It is known that the product of two T-matrices exists and is a T-matrix. On the other hand, the product of two γ-matrices may or may not exist and if it exists it may not be a γ-matrix. In the present note the author shows that the product AG of a T-matrix A and a γ-matrix G exists and is a γ-matrix and that T-matrices are the only matrices having this property for every γ-matrix. An example is given to show that the product GA is not necessarily a γ-matrix and may even fail to exist. The main theorem is followed by a discussion of the properties of the original γ-matrix G which are transferred to the product matrix AG.

J. D. Hill (East Lansing, Mich.).

Delange, Hubert. Sur la réciproque du théorème d'Abel sur les séries entières. C. R. Acad. Sci. Paris 224, 436–438 (1947).

Suppose that $s(t)$ is a complex function of the real variable t for $t\geqq 0$, that $s(0)=0$ and that $s(t)$ is of bounded variation in every finite interval $(0, L)$. Let

$$w(\lambda)=\limsup_{t\to\infty}\{\underset{t\leqq t'<\lambda t}{\text{u.b.}}|s(t')-s(t)|\}.$$

The author states the following result. If $w(\lambda)<\infty$ for some $\lambda>1$, $z_n=r_ne^{i\theta_n}$, $\lim_{n\to\infty}r_n=0$, $\lim_{n\to\infty}r_{n+1}/r_n=1$, $\limsup_{n\to\infty}|\theta_n|<\pi/2$, and if

$$\lim_{n\to\infty}\int_0^{\infty}e^{-z_nt}ds(t)=S,$$

then $\limsup_{t\to\infty}|s(t)-S|\leqq\lim_{\lambda\to1}w(\lambda)$. More generally, for any sequence t_n tending to $+\infty$,

$$\limsup_{n\to\infty}|s(t_n)-S|\leqq\lim_{\lambda\to1}w[(t_n), \lambda],$$

where

$$w[(t_n), \lambda]=\limsup_{n\to\infty}\{\underset{t_n\leqq t'<\lambda t_n}{\text{u.b.}}|s(t')-s(t_n)|\}.$$

This result generalizes the theorem of Littlewood to the effect that, if $0<\lambda_1<\lambda_2<\cdots$, $\lambda_k\to\infty$, if $\lim_{k\to\infty}\lambda_{k+1}/\lambda_k=1$, if $\sum_{k=1}^{\infty}a_ke^{-\lambda_kz}=f(z)$ converges for $\Re(z)>0$, and if $f(z)\to S$ when $z\to0$ through positive real values, then $\sum_1^{\infty}a_k$ converges to sum S provided that $|a_k|\leqq(\lambda_k-\lambda_{k-1})M/\lambda_k$ and M is constant. The author states the "one-sided" extension of his theorem which is analogous to the Hardy-Littlewood theorem in which the condition on a_k above is replaced by the weaker condition $a_k\geqq-(\lambda_k-\lambda_{k-1})M/\lambda_k$ when a_k is real.

H. R. Pitt (Belfast).

Fourier Series and Generalizations, Integral Transforms

Arbault, Jean. Sur la convergence absolue des séries trigonométriques. C. R. Acad. Sci. Paris 224, 630–631 (1947).

If P is a symmetrical perfect set in $(0, 2\pi)$ such that there exists a series $\sum\rho_n|\sin nx|$ convergent for all $x\varepsilon P$ (with $\rho_n\geqq0$, $\sum\rho_n=\infty$), then the partial sums of $\sum\rho_n|\sin nx|$ are uniformly bounded in P. Various consequences are derived, such as: if the series $\sum\rho_n|\sin nx|$ fails to converge at a single point of P, it diverges in a subset of P everywhere dense over P.

R. Salem (Cambridge, Mass.).

Zamansky, Marc. Sur l'approximation des fonctions continues. C. R. Acad. Sci. Paris 224, 704–706 (1947).

Let $f(x)$ be of period 2π and let it satisfy a Lipschitz condition of order α, $|f(x)-f(x')|\leqq M|x-x'|^{\alpha}$, $0<\alpha<1$. If $\{P_n(x)\}$ are trigonometric polynomials of order n such that $|f-P_n|\leqq KM/n^{\alpha}$, then $|P_n'(x)|\leqq AMn^{1-\alpha}/(1-\alpha)$, where A is an absolute constant. Extensions are given.

A. Zygmund (Chicago, Ill.).

Blanuša, Danilo. Der Einfluss der Unstetigkeiten einer Funktion und ihrer Ableitungen auf ihr Fouriersches Spektrum. Bull. Intern. Acad. Croate. Cl. Sci. Math. Nat. 35, 82–88 (1945).

It is known that, if the kth differential quotient of a function $f(x)$ is absolutely integrable over $(-\infty, \infty)$, then its Fourier transform $E(y)$ satisfies the relation $|y|^kE(y)\to0$ as $|y|\to\infty$. Using this result the author establishes the asymptotic behavior of $E(y)$ in case $f^{(k)}(x)$ is differentiable except at a finite number of jumps.

O. Szász.

Blanuša, Danilo. The influence of the discontinuities of a function and its derivatives on its Fourier spectrum. Rad Hrvatske Akademije Znanosti i Umjetnosti. Razred Mat.-Prirodoslov. 85, 273–285 (1942). (Croatian)

A summary is reviewed above.

Boas, R. P., Jr. Poisson's summation formula in L^2. J. London Math. Soc. 21, 102–105 (1946).

The paper establishes sufficient conditions for the validity, with a suitable interpretation, of the equation

$$\text{(1)}\quad A^{\frac12}e^{\pi abi}\sum_{n=-\infty}^{\infty}e^{2n\pi ai}G[A(n+b)]=B^{\frac12}e^{-\pi abi}\sum_{n=-\infty}^{\infty}e^{2n\pi bi}g[B(n-a)],$$

where $A>0$, $AB=2\pi$, and

$$\text{(2)}\qquad g(x)=(2\pi)^{-\frac12}\int_{-\infty}^{\infty}e^{-iux}G(u)du.$$

The main results are that, if $G(x)\varepsilon L^2(-\infty, \infty)$ and $g(x)$ is its Fourier transform (2) (defined as a limit in mean-square), then (i) the formula (1) holds for almost all points of the square $0\leqq a\leqq1$, $0\leqq b\leqq1$, both sides being convergent in mean-square with respect to (a, b); (ii) the left side is, for almost all b, convergent in mean-square (as a function of a), and the right side is, for almost all a, convergent in mean-square (as a function of b); (iii) "convergent in mean-square" can be replaced in (ii) by "summable (C, δ), $\delta>0$, almost everywhere." If $G(x)\varepsilon L^2(-\infty, \infty)$ and is even or odd, nonincreasing for $x>0$, and continuous, then (1) holds for all b and for $a\not\equiv0 \pmod{2\pi}$, the left side being convergent and the right side being summable (C, δ), $\delta>0$. Certain generalisations of these results are also proved.

E. H. Linfoot (Bristol).

✶Steffensen, J. F. Bounds of certain trigonometrical integrals. C. R. Dixième Congrès Math. Scandinaves 1946, pp. 181–186. Jul. Gjellerups Forlag, Copenhagen, 1947.

Let $f(t)$ be strictly decreasing, $0\leqq\varphi(t)\leqq1$, $\lambda=\int_a^b\varphi(t)dt$, $0<\lambda<1$. The author has given the inequality

$$\int_{b-\lambda}^{b}f(t)dt<\int_a^bf(t)\varphi(t)dt<\int_a^{a+\lambda}f(t)dt$$

[Skand. Aktuarietidskr. 1, 82–97 (1918)]. Here he applies it to deduce the following results. If $f(t)\downarrow 0$ as $t\uparrow\infty$,

$$\sum_{\nu=1}^{\infty}(-1)^{\nu}f(\nu\pi)<\int^{\infty}f(t)\cos t\,dt<\sum_{\nu=0}^{\infty}(-1)^{\nu}f(\nu\pi),$$

$$\sum_{\nu=0}^{\infty}(-1)^{\nu}f((\nu+\tfrac{1}{2})\pi)<\int_0^{\infty}f(t)\sin t\,dt$$

$$<f(0)+\sum_{\nu=0}^{\infty}(-1)^{\nu}f((\nu+\tfrac{1}{2})\pi).$$

He gives more precise inequalities in terms of $g(x)=\int_x^{x+1}f(t)dt$.

R. P. Boas, Jr. (Providence, R. I.).

✱Petersen, Richard. On Lerch's theorem. C. R. Dixième Congrès Math. Scandinaves 1946, pp. 376–383. Jul. Gjellerups Forlag, Copenhagen, 1947.

By use of Blaschke products the author proves that a Laplace integral which vanishes on a real arithmetical progression vanishes identically. *H. Pollard.*

Amerio, Luigi. Sull'inversione della trasformata di Laplace. Rend. Accad. Sci. Fis. Mat. Napoli (4) 10, 232–259 (1940).

If $f(p)=\int_{-\infty}^{\infty}e^{-pt}F(t)dt$, $\alpha<\Re(p)<\beta$, where $F(t)$ is integrable in every finite interval, then

$$F(t)=\lim_{\lambda\to\infty}(2\pi i)^{-1}\int_{k-i\Omega}^{k+i\Omega}e^{pt}f(p)(1+p^2/\lambda^2)^n dp,$$

$\alpha<k<\beta$, n a positive integer, for almost every t, uniformly in an interval of continuity, etc. Similar results are given for the derivatives of $F(t)$, if they exist.

R. P. Boas, Jr. (Providence, R. I.).

Widder, D. V. Green's functions for linear differential systems of infinite order. Proc. Nat. Acad. Sci. U. S. A. 33, 31–34 (1947).

If $0<c<a_1<a_2<\cdots$, $\sum a_k^{-2}<\infty$, and

$$E(s)=s\prod_{k=1}^{\infty}(1-s^2/a_k^2),\quad G(x)=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\{e^{sx}/E(s)\}ds,$$

the function $y(x)=\int_{-\infty}^{\infty}G(x-t)\varphi(t)dt$ is a solution of the differential system

$$E(D)y(x)=\varphi(x),\quad y(-\infty)=0,\quad y(+\infty)=\int_{-\infty}^{\infty}\varphi(t)dt,$$

where D is the operation of differentiation with respect to x and $\varphi(x)$ is continuous and absolutely integrable in $(-\infty,\infty)$. *C. Miranda* (Naples).

in Library

✱Titchmarsh, E. C. Eigenfunction Expansions Associated with Second-Order Differential Equations. Oxford, at the Clarendon Press, 1946. 175 pp. $7.00.

The object of this book is the study by complex variable methods of the expansion problems connected with the equation (*) $y''+\{q(x)-\lambda\}y=0$. A considerable part of the material has already appeared [J. London Math. Soc. 14, 274–278 (1939); Quart. J. Math., Oxford Ser. 11, 129–145 (1940); 12, 33–50, 89–107, 154–166 (1941); 16, 103–114 (1945); these Rev. 1, 56; 2, 53; 3, 121, 235; 7, 247]. Chapter I treats the classical Sturm-Liouville case by the calculus of residues. Chapter II studies the simplest singular case, in which the spectrum is discrete. Generality is reached in chapter III, where the spectrum can be of a mixed character; in this case the expansions are no longer given by infinite series, but by improper Stieltjes integrals. Precise conditions for their validity are too complicated to state here. Chapter IV illustrates the preceding material by a variety of examples including expansions in terms of the classical orthogonal polynomials, Fourier-Bessel series, and Hankel's integrals.

Chapter V concerns the relation between the spectrum and the behavior of the function $q(x)$ of (*). Suppose, for example, that the interval is $(0,\infty)$. There are essentially four cases, as follows. (Actually further conditions are needed, so that the classification is not complete, but all ordinary problems are included.) If $q(x)\to\infty$ as $x\to\infty$ there is a pure point spectrum. If $q(x)\to 0$, there is a continuous spectrum in $(0,+\infty)$, with a point spectrum (possibly missing) in $(-\infty,0)$. If $q(x)\to-\infty$, but so that $\int^{\infty}|q(x)|^{-\frac{1}{2}}dx$ diverges, the spectrum extends continuously from $-\infty$ to ∞. Finally if $q(x)\to-\infty$, but the integral converges, there is a continuous spectrum in $(-\infty,0)$, and a point-spectrum (possibly missing) in $(0,\infty)$. The remaining chapters concern a variety of topics: alternative conditions for validity of the expansions, the distribution of eigenvalues, the BWK method and conditions for the summability of the expansions. *H. Pollard* (Ithaca, N. Y.).

Sarginson, Kathleen. An expansion in eigenfunctions. J. London Math. Soc. 21, 147–157 (1946).

The author studies the expansion problems for the operator $(L-\lambda)\varphi=0$, where

$$(*)\qquad L=\frac{d^2}{dx^2}+2\tanh x\frac{d}{dx}+\frac{l(l+1)}{\cosh^2 x}.$$

This operator arises in the solution of the Klein-Gordon wave equation for a free electron. Here l is a positive integer and λ is a constant to be determined so that the solutions of (*) are to be quadratically integrable over (i) $-\infty<x<\infty$, or (ii) $0<x-\frac{1}{2}\pi i<\infty$, or (iii) $-\infty<x-\frac{1}{2}\pi i<0$. The methods are those developed by Titchmarsh in a recent series of papers, and which have since been incorporated into the book reviewed above. *H. Pollard* (Ithaca, N. Y.).

✱Hilding, Sven H. Sur des systèmes complets dans l'espace de Hilbert. C. R. Dixième Congrès Math. Scandinaves 1946, pp. 290–292. Jul. Gjellerups Forlag, Copenhagen, 1947.

A review of results published earlier [Ark. Mat. Astr. Fys. 32B, no. 7 (1945); these Rev. 8, 151]. *H. Pollard.*

Ghizzetti, Aldo. Sull'approssimazione delle funzioni continue. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 80, 225–230 (1945).

Let $\{\Phi_k(x)\}$ be a sequence of absolutely continuous functions whose derivatives $\{\varphi_k(x)\}$ are complete in $L^2(a,b)$. Let the graph of $\gamma(x)$ be a polygonal line with $n+1$ vertices. Then for sufficiently large p there is a sum

$$g_p(x)=c+\sum_{k=0}^{n+p-1}c_k\Phi_k(x)$$

whose graph passes through the vertices of $\gamma(x)$ and which minimizes $\int_a^b\{g_p'(x)\}^2dx$; $\lim_{p\to\infty}g_p(x)=\gamma(x)$ uniformly in (a,b). *R. P. Boas, Jr.* (Providence, R. I.).

Conzelmann, Rolf. Beiträge zur Theorie der singulären Integrale bei Funktionen von mehreren Variablen. I. Comment. Math. Helv. 19, 279–315 (1947).

The theory of representation of functions by "singular integrals," due primarily to Lebesgue [Ann. Fac. Sci. Univ.

Toulouse (3) 1, 25–117 (1909)] is concerned with the conditions on the "kernel" $K_n(x, \xi)$ under which for certain classes of functions f we have the formula

$$(*) \qquad f(x) = \lim_{n\to\infty} \int_a^b f(\xi) K_n(x, \xi) d\xi.$$

Similarly we can consider the problem [Hahn, Denkschr. Akad. Wiss. Wien 93, 585–692 (1916)] of implying

$$(**) \qquad f'(x) = \lim_{n\to\infty} \int_a^b f(\xi) \frac{\partial}{\partial x} K_n(x, \xi) d\xi.$$

An extension of (*) to the case of functions f of several variables does not involve new ideas [Camp, Trans. Amer. Math. Soc. 14, 42–64 (1913)]. In the present paper the author is concerned with extensions of (**) to several variables. The results are not simple enough to be stated here. Suffice it to say that, for the extension of (**), the existence of the first partial derivatives of f is of little consequence and one has to employ stronger conditions, like the existence of a differential. *A. Zygmund* (Chicago, Ill.).

Kovanko, A. S. Sur la correspondence entre les classes diverses de fonctions presque-périodiques généralisées. Bull. [Izvestiya] Math. Mech. Inst. Univ. Tomsk 3, 1–36 (1946). (Russian. French summary)

This paper presents a systematic discussion of different definitions of pseudoperiodicity of functions. The author introduces appropriate definitions of measure of a set on the infinite axis and of distance of two functions and derives from them in the usual way the corresponding definitions of pseudoperiodicity. By means of the different definitions of measure he obtains three different definitions of uniformly summable functions. By combination of three classes of pseudoperiodic functions and three classes of uniformly summable functions he obtains nine classes of pseudoperiodic functions. He constructs examples for them. A paragraph is concerned with theorems on the pseudoperiodicity of functions which are limits (in different topologies) of sequences of certain types of pseudoperiodic functions. Finally one paragraph deals with composite functions and determines the pseudoperiodic class of functions of pseudoperiodic functions of different types. *František Wolf.*

✶Bohr, Harald. On some functional spaces. C. R. Dixième Congrès Math. Scandinaves 1946, pp. 313–319. Jul. Gjellerups Forlag, Copenhagen, 1947.

Brief exposition of results of Bohr and Følner [Acta Math. 76, 31–155 (1945); these Rev. 7, 154] and of Følner [Copenhagen thesis, 1944; these Rev. 8, 151], concerning almost periodic functions.

Polynomials, Polynomial Approximations

Ibraghimoff, I. Sur la valeur asymptotique de la meilleure approximation d'une fonction ayant un point singulier réel. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 429–460 (1946). (Russian. French summary)

Proofs of some theorems on the best approximation to the function $f(x) = (a-x)^s\{\log(a-x)\}^m$ which the author previously published without proof [C. R. (Doklady) Acad. Sci. URSS (N.S.) 49, 238–240 (1945); these Rev. 8, 153]. The theorems for the case when m is nonintegral are given in a slightly different form. *A. C. Offord.*

Bernstein, S. Complément au travail de I. Ibraghimoff "Sur la valeur asymptotique de la meilleure approximation d'une fonction ayant un point singulier réel." Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 10, 461–462 (1946). (Russian. French summary)

The author generalizes some of the results of Ibraghimoff [see the preceding review] in the following way. Suppose $\Phi_1(x)$ is regular and satisfies $|\Phi_1(x)| \leq M$ in an ellipse with foci at the points -1 and $+1$ and such that the sum of its semi-axes is equal to R. Suppose, furthermore, that $R > \rho = c + (c^2-1)^{\frac{1}{2}}$ and that $E_n[f(x)]$ denotes the best approximation in $(-1, 1)$ to $f(x)$ by polynomials of degree n. Then

$$E_n[(c-x)^{m+1}\Phi_1(x)] < (2MR(R-\rho)^{-1} + \epsilon_n) E_n[(c-x)^{m+1}],$$

where $\epsilon_n \to 0$ as $n \to \infty$. He extends this result further by replacing $(c-x)^{m+1}$ by an arbitrary function $\Phi_0(x)$ for which $\lim \{E_n[\Phi_0(x)]\}^{1/n} = \rho < R$. *A. C. Offord.*

✶Merli, Luigi. Sulla convergenza degli integrali dei polinomi di interpolazione di Hermite per un particolare sistema di punti interpolanti. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 168–176. Edizioni Cremonense, Rome, 1942.

Let $S_n(x)$ be the Hermite polynomial of interpolation of degree $2n-1$, associated with a given function $f(x)$ defined in $(-1, 1)$ and with the abscissas $\cos\{2k\pi/(2n+1)\}$, $k = 1, \cdots, n$. Refining results of the reviewer [Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publ., v. 23, New York, 1939, p. 333; these Rev. 1, 14] the author proves that

$$\lim_{n\to\infty} \int_{-1}^{1} |f(x) - S_n(x)| dx = 0$$

provided that $f'(x)$ is continuous in $-1 \leq x \leq 1$. *G. Szegö*

✶Belardinelli, Giuseppe. Su una espressione asintotica dei polinomi di Hermite. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 105–111. Edizioni Cremonense, Rome, 1942.

The author uses Perron's asymptotic representation of confluent hypergeometric functions to obtain asymptotic representations of Hermite polynomials. Neither the method nor the results seem to be essentially new.

A. Erdélyi (Pasadena, Calif.).

Koschmieder, Lothar. Beispiele des Gebrauchs gewisser Ableitungsformeln von Liouville, Spitzer und Schlömilch. Revista Mat. Hisp.-Amer. (4) 6, 240–248 (1946).

The derivative formula

$$(1) \qquad d^h z / d(x^{-1})^h = (-1)^h x^{h+1} d^h (x^{h-1} z)/dx^h$$

[attributed to Spitzer and Schlömilch], and some formulas of Liouville, for example,

$$(2) \qquad \frac{d^h z}{d\xi^h} = 2^h \frac{d^{(h-1)/2}}{dx^{(h-1)/2}} \left\{ x^{h/2} \frac{d^{(h+1)/2} z}{dx^{(h+1)/2}} \right\},$$

where h is odd and $\xi = x^{\frac{1}{2}}$, are extended to the case of more than one independent variable. These formulas and their extensions are then used to obtain a variety of formulas concerning such polynomials (in one or more variables) as are expressible by means of derivatives: for example, the polynomials of Gegenbauer, Jacobi, Hermite, Laguerre.

Most of the examples are not new but the method provides simple machinery for their derivation. *I. M. Sheffer*

Special Functions

Parodi, Maurice. Sur une relation satisfaite par la fonction Γ. Ann. Soc. Sci. Bruxelles. Sér. I. 60, 200–201 (1946).

The relation established should be $n!\sum_{k=0}^{n}\Gamma(p-k)/(n-k)! = p\Gamma(p)/(p-n)$. [The author writes $\sum_{k=1}^{n}$ instead of $\sum_{k=0}^{n}$.] The result is known. The author's method of derivation is by symbolic calculus. It may be remarked that

$$n!\sum_{k=0}^{n}\frac{\Gamma(p-k)}{(n-k)!}=\frac{n!}{p-n}\sum_{k=0}^{n}\frac{(p-k)\Gamma(p-k)-(n-k)\Gamma(p-k)}{(n-k)!}$$

$$=\frac{n!}{p-n}\sum_{k=0}^{n}\left\{\frac{\Gamma(p-k+1)}{(n-k)!}-\frac{\Gamma(p-k)}{(n-k-1)!}\right\}=\frac{\Gamma(p+1)}{p-n}.$$

S. C. van Veen (Delft).

Mitra, S. C. On certain infinite integrals involving Struve functions and parabolic cylinder functions. Proc. Edinburgh Math. Soc. (2) 7, 171–173 (1946).

Part of the present note is based on the formula

$$\int_0^\infty e^{-x^2/4}x^m D_n(x)dx=\frac{\pi^{\frac{1}{2}}2^{\frac{1}{2}(n-m-1)}\Gamma(m+1)}{\Gamma(\frac{1}{2}m-\frac{1}{2}n+1)},\quad \Re(m)>-1,$$

[Whittaker and Watson, A Course of Modern Analysis, 3d ed., 1920, p. 353]. From this, the author deduces, among other results, the following, in which s is a positive integer:

$$\int_0^\infty (xy)^{\frac{1}{2}}x^s e^{-x^2/4}H_{s+\frac{1}{2}}(xy)D_{2s+2}(x)dx=(-1)^s y^{s+1}e^{-y^2/4}D_{2s+1}(y),$$

$$\int_0^\infty e^{-x^2/4}D_{2s}(x)\cos(xy)dx=(\tfrac{1}{2}\pi)^{\frac{1}{2}}(-1)^s y^{2s}e^{-y^2/2},$$

$$\int_0^\infty x^{s-\frac{1}{2}}e^{-x^2/4}K_{s-\frac{1}{2}}(xz)D_{2s}(x)dx$$
$$=(\tfrac{1}{2}\pi)^{\frac{1}{2}}(-1)^s\Gamma(2s)z^{s-\frac{1}{2}}e^{z^2/4}D_{-2s}(z),$$

$$\int_0^\infty x^{s+\frac{1}{2}}e^{-x^2/4}K_{s+\frac{1}{2}}(xz)D_{2s+1}(x)dx$$
$$=(\tfrac{1}{2}\pi)^{\frac{1}{2}}(-1)^s\Gamma(2s+3)z^{s-\frac{1}{2}}e^{z^2/4}D_{-2s-3}(z).$$

Other results are obtained through the use of the Laplace transformation. *M. A. Basoco* (Lincoln, Neb.).

Shanker, Hari. On the Hankel transform of generalized hypergeometric functions. J. London Math. Soc. 21, 194–198 (1946).

The author obtains the following result regarding the Hankel transforms of the hypergeometric function ${}_2F_2$. If

$$\varphi(x)=\frac{x^{2\lambda-\nu-\frac{1}{2}}}{2^{\lambda-1}\Gamma(\gamma)\Gamma(\lambda)}\,{}_2F_2[\alpha,\beta;\gamma,\lambda;-\tfrac{1}{2}x^2],$$

$$\psi(x)=\frac{\Gamma(\beta-\alpha)x^{\nu-2\lambda+2\alpha+\frac{1}{2}}}{\Gamma(\gamma-\alpha)\Gamma(\beta)\Gamma(\nu-\lambda+\alpha+1)2^{\nu-\lambda+\alpha}}$$
$$\times{}_2F_2[\alpha,\alpha-\gamma+1;\alpha-\beta+1,\nu-\lambda+\alpha+1;-\tfrac{1}{2}x^2]$$
$$+\frac{\Gamma(\alpha-\beta)x^{\nu-2\lambda+2\beta+\frac{1}{2}}}{\Gamma(\gamma-\beta)\Gamma(\alpha)\Gamma(\nu-\lambda+\beta+1)2^{\nu-\lambda+\beta}}$$
$$\times{}_2F_2[\beta,\beta-\gamma+1;\beta-\alpha+1,\nu-\lambda+\beta+1;-\tfrac{1}{2}x^2],$$

then $\varphi(x)$ and $\psi(x)$ are Hankel transforms of each other of order ν, provided that $\Re(\lambda)>0$, $\Re(\nu-\lambda+\alpha+1)>0$, $\Re(\nu-\lambda+\beta+1)>0$ and that

$$\tfrac{3}{2}<\Re(2\gamma-\tfrac{1}{2})<\Re(2\lambda-\nu)<\Re(2\beta+\tfrac{1}{2})<\Re(2\alpha+\tfrac{1}{2}).$$

A number of applications to confluent forms are discussed. *N. A. Hall* (East Hartford, Conn.).

McLachlan, N. W. Mathieu functions of fractional order. J. Math. Phys. Mass. Inst. Tech. 26, 29–41 (1947).

The author starts from Mathieu's differential equation and from its associated form in which the independent variable is purely imaginary. Their solutions of fractional order are written as Fourier expansions, the coefficients of which are determined by recurrence relations. These functions are then represented by Bessel function product series, again determining the coefficients by recurrence relations. The equivalence of both representations for functions of definite type is established. Then Bessel function series are considered as solutions of the differential equation using similar procedures in the determination of coefficients. These solutions are identified with definite solutions of Fourier type. Asymptotic expressions for large values of the independent variable in the associated case are established for the different types of functions. Finally integral equations are given which are satisfied by Mathieu functions of definite types and the degeneration of Mathieu functions to Bessel functions of equal order is discussed. *M. J. O. Strutt* (Eindhoven).

McLachlan, N. W. Corrections to paper "Mathieu functions and their classification." J. Math. Phys. Mass. Inst. Tech. 26, 78 (1947).

The paper appeared in the same J. 25, 209–240 (1946); these Rev. 8, 156.

Byuler, G. A. On the integral representation of Mathieu functions. Bull. [Izvestiya] Math. Mech. Inst. Univ. Tomsk 3, 191–197 (1946). (Russian)

The author uses the most general form of the integral equation satisfied by Mathieu functions to represent Mathieu functions as contour integrals which are generalizations of the well-known contour integrals for the Hankel and Bessel functions. *F. Smithies.*

Harmonic Functions, Potential Theory

Lelong, Pierre. Sur une propriété simple des polynomes. C. R. Acad. Sci. Paris 224, 883–885 (1947).

Let E be a closed set in the z-plane whose complement is simply connected and let z_0 be a boundary point of E. We consider all polynomials $P_n(z)$ for which $|P_n(z)|\leqq 1$ holds on E. A necessary and sufficient condition that to each ϵ a neighborhood $\Omega(z_0,\epsilon)$ of z_0 belongs such that $|P_n(z)|\leqq(1+\epsilon)^n$ in Ω is that z_0 is a regular point; n is the degree of $P_n(z)$. *G. Szegö* (Stanford University, Calif.).

Leja, François. Un critère de régularité des points-frontière dans le problème de Dirichlet plan. C. R. Acad. Sci. Paris 224, 882–883 (1947).

Let F be the boundary of a domain D of the complex plane having the point at infinity as an interior point; the

transfinite diameter of F is assumed positive. The author states that a necessary and sufficient condition for a point z_0 of F to be regular is that for every sequence of polynomials $\{P_n(z)\}$ uniformly bounded on F and every $\epsilon>0$ the sequence $\{(1+\epsilon)^{-n}P_n(z)\}$ is uniformly bounded in some neighborhood $|z-z_0|<r$ of z_0. The value of r depends in general on $\{P_n(z)\}$ and on ϵ. Brief indications of the derivation of this result are given. *F. W. Perkins* (Hanover, N. H.).

Walsh, J. L. On the location of the critical points of harmonic measure. Proc. Nat. Acad. Sci. U. S. A. **33**, 18–20 (1947).

Let $\alpha=\alpha_1+\cdots+\alpha_n$ be a set of mutually disjoint arcs on the circle C ($|z|=1$). The critical points of the harmonic measure of α with respect to the interior of C lie in a region bounded by n arcs of circles orthogonal to C.

J. Ferrand (Caen).

Papaspyros, Anast. Introduction to the theory of harmonic measure of R. Nevanlinna and its application to the solution of the Dirichlet problem. Bull. Soc. Math. Grèce **21**, 58–66 (1941). (Greek)

Kappos, D. A. On the Dirichlet problem. Bull. Soc. Math. Grèce **21**, 104–125 (1941). (Greek)

Expository lectures.

Ridder, J. Über harmonische Funktionen. Nieuw Arch. Wiskunde (2) **22**, 162–170 (1946).

It is shown that the finite-valued function $u(x, y)$ is harmonic in its (x, y)-domain of definition D provided it has finite first partial derivatives, with respect to x and y, which are summable over every rectangular area J in D having boundary sides parallel to the coordinate axes, which are linearly summable with respect to x and y, and for which the rectangle function $\Phi(J)\equiv\int_R u_y dx-u_x dy$, where R is the boundary of J, vanishes identically in D.

E. F. Beckenbach (Los Angeles, Calif.).

Reade, Maxwell O. On functions having subharmonic logarithms. Bull. Amer. Math. Soc. **53**, 89–95 (1947).

Dans le plan le laplacien généralisé de Privaloff s'obtient en multipliant par $8/r^2$ la différence entre la moyenne sur l'aire du cercle de rayon r et la valeur de la fonction u au centre, puis passant à la limite. Saks a signalé que, presque partout, pour u sousharmonique, cette limite existe et vaut la densité de la distribution de masses associée [Rec. Math. [Mat. Sbornik] N.S. **9**(51), 451–456 (1941); ces Rev. **2**, 366]. L'auteur donne un résultat analogue pour $f(x, y)>0$ de logarithme sousharmonique; il remplace la différence indiquée par celle du carré de la moyenne périphérique et de la moyenne de f^2 sur l'aire; l'extension se fait avec la densité de la distribution associée à $\log f$ et en la multipliant par f^2. *M. Brelot* (Grenoble).

Reade, Maxwell O. A particular generalized Laplacian. Bull. Amer. Math. Soc. **53**, 96–97 (1947).

En extension de résultats antérieurs, l'auteur démontre: si $F(t)$ admet $F'(t)$ continue, positive; si $f(x, y)$ a des dérivées premières continues, et si $F\{\alpha x+\beta y+f(x, y)\}$ ou si $F\{\log[(x-\alpha)^2+(y-\beta)^2]+f(x, y)\}$ est sousharmonique, alors f est sousharmonique. L'auteur utilise un laplacien généralisé périphérique. *M. Brelot* (Grenoble).

Gabriel, R. M. A note upon functions positive and subharmonic inside and on a closed convex curve. J. London Math. Soc. **21**, 87–90 (1946).

The author proves that, if $u(z)\geqq 0$ is subharmonic inside and on a closed convex curve Γ and if z_0 is a point inside Γ at a distance l from Γ, then $u(z_0)\leqq(\pi l)^{-1}\int_\Gamma u(z)|dz|$. It is shown that the multiplicative constant is the best possible and that the convexity is necessary for any inequality of this form. *František Wolf* (Berkeley, Calif.).

Nicolesco, Miron. Remarque sur le potentiel newtonien. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași] **1**, 256–258 (1946).

Let M be a distribution of mass on a bounded domain D and let the density $\mu(x, y, z)$ of M be continuous in D. Then the potential $V(x, y, z)$ of M satisfies Poisson's equation $\Delta V=-4\pi\mu(x, y, z)$ in D. The purpose of this note is to show the contribution of the derivatives V_{xx}, V_{yy} and V_{zz} to Poisson's equation. Indeed, let Σ_ρ be a sphere lying in D with center (x, y, z) and radius ρ and let $V^*(x, y, z)$ be the potential of that part of M that lies outside Σ_ρ. Then by a modification of a familiar derivation of Poisson's equation [O. D. Kellogg, Foundations of Potential Theory, Springer, Berlin, 1929], the author shows that

$$\partial^2 V/\partial x^2=\partial^2 V/\partial x_*^2-(4/3)\pi\mu(x, y, z),$$

where $\partial^2 V/\partial x_*^2=\lim_{\rho\to 0}\partial^2 V^*/\partial x^2$; similar results hold for the other two derivatives. *M. O. Reade* (Ann Arbor, Mich.).

Castoldi, L. Sul potenziale di un doppio strato non uniforme. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 1051–1054 (1946).

The author shows that the field due to a magnetic shell of nonuniform strength S is equal to the sum of fields due to (i) a linear current along the boundary of the shell, (ii) linear currents along the lines of discontinuity of S, (iii) surface currents between the lines of constant S.

I. Opatowski (Ann Arbor, Mich.).

Fichera, Gaetano. Decomposizione al modo di Poincaré delle funzioni bi-iperarmoniche in due variabili. Rend. Accad. Sci. Fis. Mat. Napoli (4) **11**, 134–149 (1941) = Ist. Naz. Appl. Calcolo (2) no. **113**.

Let D be a simply connected region of the plane and let B_i, $i=1, \cdots, n$, be mutually disjoint continua in D. If u is harmonic, Green's identity yields the decomposition formula for u: $u(P)=\sum_{i=0}^n u_i(P)+\sum_{i=1}^n k_i\log\overline{O_iP}$, where $u_0(P)$ is harmonic in D; $u_i(P)$, $i=1, \cdots, n$, is harmonic outside B_i and vanishes at infinity; O_i is any point of B_i and k_i is a constant. The author generalizes this formula to the case where u is biharmonic, i.e., a solution of $\Delta^4 u=0$. The result is:

$$u(P)=\sum_{i=0}^n u_i(P)+\sum_{i=1}^n\{a_i\overline{O_iP}^2+b_i(x-x_i)+c_i(y-y_i)\}\log\overline{O_iP}$$
$$+\sum_{i=1}^n\{d_i\cos 2(\widehat{x}, \overline{O_iP})+e_i\sin 2(\widehat{x}, \overline{O_iP})\}.$$

Here u_0 is biharmonic in D; u_i, $i=1, \cdots, n$, is biharmonic outside B_i; $O_i=(x_i, y_i)$ is any point of B_i. The constants a_i, etc., are given by formulas involving integrals of u and its derivatives over suitable curves in D. The proofs are made, just as in the harmonic case, with the use of Green's identity. Here the function $\overline{OP}^2\log\overline{OP}$ plays the role which in the harmonic case is played by $\log\overline{OP}$.

J. W. Green (Los Angeles, Calif.).

Differential Equations

Di Bello, Maria. Un'equazione analoga a quella di Clairaut dedotta della geometria di Lobaceschi. Rend. Accad. Sci. Fis. Mat. Napoli (4) 10, 111–114 (1940).

Di Bello, Maria. Inviluppi di curve piane ed equazioni di Clairaut generalizzate. Rend. Accad. Sci. Fis. Mat. Napoli (4) 10, 281–287 (1940).

Clairaut's equation in the theory of ordinary differential equations has the form $y=px+f(p)$, $p=dy/dx$. This equation is associated with the family of straight lines $y=ax+b$, $b=f(a)$. The author begins with the family $y=ax+b$, $\phi(a,b)=0$, and obtains by eliminating a and b a generalized Clairaut equation $\phi(y', y-xy')=0$. If one begins with more complicated equations, such as $f(x,y)-ax-b=0$, $\phi(a,b)=0$, more complicated equations of Clairaut type are obtained. The first paper is concerned with some equations derived from consideration of envelopes in Lobachevskian geometry.

R. Bellman (Princeton, N. J.).

Vâlcovici, Victor. Sur le mouvement d'un solide dans un milieu résistant. Disquisit. Math. Phys. 1, 93–100 (1940).

The equation $Mdv/dt=A+Bv+Cv^2$, with M, A, B, C constant, is completely solved.

P. Franklin.

Chiellini, Armando. Gli invarianti differenziali dell'equazione $y'=c_0y^3+3c_1y^2+3c_2y+c_3$. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 4(73), 227–247 (1940).

Equation (1) $Y'=c_0Y^3+3c_1Y^2+3c_2Y+c_3$, with $c_i=c_i(x)$, under change of dependent variable $Y=y-c_1/c_0$, is transformed into (2) $y'=p_0y^3+3p_2y+p_3$, with $p_i=p_i(x)$. Equation (2) is subjected to transformations of the type (3) $y=\lambda(x)\cdot z$, $\xi=\phi(x)$, where $\lambda(x)$, $\phi(x)$ are arbitrary functions with continuous first derivatives and $\lambda\phi'\neq 0$, introducing a new independent variable ξ and a new dependent variable z. Let $J=D(x,y)/D(\xi,z)=\lambda/\phi'$. The transformed equation is (4) $dz/d\xi=q_0z^3+3q_2z+q_3$ (with simple formulas for q_0, q_2, q_3), of the same type as (2). Any function $f(p_0,p_2,p_3)$ such that $f(q_0,q_2,q_3)=J^nf(p_0,p_2,p_3)$ under any transformation of type (3) is called a relative invariant of weight n. If $n=0$, f is called an absolute invariant. The author introduces three relative invariants: $s_3=p_3p_0^2$, of weight 3; $s_1=p_0\exp(3\int p_2dx)$, of weight 1; and $s_5=-p_0s_3'+3s_3(p_0'+3p_0p_2)$, of weight 5, giving him two absolute invariants: $S_1=s_3/s_1^3$ and $S_2=s_5/s_1^5$. We note that s_1 is determined with an arbitrary constant multiplier, say $1/k$, while s_3 and s_5 are completely determined, so that S_1 and S_2 are determined with constant multipliers k^3 and k^5, respectively. The functions q_0 and q_2 in (3) are easily made to be 1 and 0, respectively, by proper choice of λ, ϕ, and in this case q_3 turns out to be a determination of S_1 for the variables z, ξ resulting. Thus (4) takes the form $dz/d\xi=z^3+S_1$ $[S_1=S_1(\xi)]$.

The author shows how to express p_0, p_2, p_3 in terms of s_1, s_3, s_5 and their first derivatives. Thus any relation not involving derivatives, for either set, yields a relation, with or without first derivatives, for the other set. Also, any relative invariant must be expressible as a function of s_1, s_3, s_5 and their first derivatives. As an application of the property mentioned just above, the author determines a necessary and sufficient condition that (2) can be put in the form $dz/d\xi=z^3+\xi^n$. For this equation, he finds that $S_2^n=(-n)^nS_1^{n-1}$, and concludes that that is the condition. Owing to the arbitrary constant multipliers, this condition should be corrected to read $S_2^n=hS_1^{n-1}$ (h constant). However, this does not really seem to be an illustration of the principle, and it seemed to the reviewer that the converse proof, omitted by the author, had to be carried through.

The author then proves that a necessary and sufficient condition that (2) can be transformed, by formulas involving quadratures, into an equation with constant coefficients (hence with variables separable) is that $S_2^3=aS_1^5$ with a constant. The method for determining the transformation is given.

Finally the author shows how certain other types of differential equations can be transformed into form (1), so that the theory can be applied to them.

A. B. Brown.

Germay, R. H. J. Sur une méthode d'intégration par approximations successives des équations différentielles de forme normale. Ann. Soc. Sci. Bruxelles. Sér. I. 61, 18–26 (1947).

The paper deals with a method of successive approximations for differential equations of the form $y'=A(x)y+\phi(x,y)$. The method is equivalent to first making the variation of constants, $y=u\exp(\int^xA(t)dt)$, and then using the usual successive approximations on the resulting differential equation for u.

P. Hartman (Baltimore, Md.).

Fréchet, Maurice. Compléments à certains théorèmes d'existence et d'unicité des solutions des équations différentielles. Ann. Chaire Phys. Math. Kiev 4, 207–241 (1939). (Ukrainian and French)

The author makes some elementary remarks on the uniqueness criterion of Osgood for solutions of an ordinary differential equation and on Montel's selection theorem for solutions of a sequence of differential equations. As an application of these latter remarks, the author shows that, if α, β, γ, δ are real constants such that $(\alpha-\delta)^2+4\beta\gamma<0$; $f(x,y)$, $g(x,y)$ real continuous functions of (x,y) in a vicinity of $(x,y)=(0,0)$ satisfying $f(x,y)=o(x^2+y^2)^{\frac{1}{2}}$ and $g(x,y)=o(x^2+y^2)^{\frac{1}{2}}$ as $x^2+y^2\rightarrow 0$; $r=r(\theta,\lambda)$ a solution, in polar coordinates, of the differential equation

$$(\alpha x+\beta y+f(x,y))dy=(\gamma x+\delta y+g(x,y))dx$$

satisfying the initial condition $r(\theta_0,\lambda)=\lambda r_0$, where $r_0>0$; finally, $r=R(\theta)$ the solution of the corresponding unperturbed differential equation (with $f(x,y)\equiv g(x,y)\equiv 0$) satisfying the initial condition $R(\theta_0)=r_0$, then $r(\theta,\lambda)/\lambda$ tends uniformly to $R(\theta)$ on every finite θ-interval as $\lambda\rightarrow+0$.

P. Hartman (Baltimore, Md.).

✶Caligo, Domenico. Un criterio sufficiente di stabilità per le soluzioni dei sistemi di equazioni integrali lineari e sue applicazioni ai sistemi di equazioni differenziali lineari. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 177–185. Edizioni Cremonense, Rome, 1942.

The author considers the problem of the boundedness of solutions of the equation (1) $y'=(A(t)+B(t))y+f(t)$, where y and $f(t)$ are n-dimensional column vectors and A and B are $n\times n$ matrices. Let the norms of y and A be denoted by $\|y\|$ and $\|A\|$ and be, respectively, the sums of the absolute values of their components and elements. Then the following result is obtained. If $\int^\infty\|B(t)\|dt<\infty$, $\int^\infty\|f(t)\|dt<\infty$ and $\|Y(t)Y^{-1}(t_1)\|\leqq c_1$, c_1 independent of t and t_1; $t, t_1\geqq t_0$, where $Y(t)$ is defined by $Y'=A(t)Y$, $Y(0)=I$, then all solutions of (1) are bounded as $t\rightarrow\infty$. The proof is obtained by transforming the differential equation into a Volterra integral equation. Other related results are also obtained.

R. Bellman (Princeton, N. J.).

Wintner, Aurel. (L^2)-connections between the potential and kinetic energies of linear systems. Amer. J. Math. 69, 5–13 (1947).

The author considers the differential equation $x''+f(t)x=g(t)$ with t ranging over $(0, \infty)$. If (a) $f(t)=O(1)$ as $t\to\infty$, or more generally if (b) $f(t)\leqq c<\infty$, $t\to\infty$, then the author shows that if x and $g(t)$ are in $L^2(0, \infty)$ then so is $x'(t)$. [For case (a) the result follows at once from theorem 260 of "Inequalities" by Hardy, Littlewood, and Pólya [Cambridge University Press, 1934], where the proof is the same as that of the author. The result is also a special case of the general nth order theorems of Halperin [Ann. of Math. (2) 38, 880–919 (1937)] and Halperin and Pitt [Duke Math. J. 4, 613–625 (1938)], where the L^p case of nth order systems is considered.]

For case (a) the differential equation shows that x'' is in L^2 from which it is clear that x', being in L^2, must tend to zero as $t\to\infty$; this is a result the author proves somewhat differently. By considering the Wronskian the author shows easily for the case (a) that any solution of the homogeneous equation linearly independent of a solution in L^2 cannot be bounded as $t\to\infty$. Applications are made to a problem of interest in spectral theory. *N. Levinson.*

Péyovitch, Tadya. Sur la valeur à l'infini des intégrales de certaines équations différentielles. Revue Sci. 84, 354–356 (1946).

In the system

$$\frac{dz_s}{dt}=a_{s1}e^{z_1}+a_{s2}e^{z_2}+\cdots+(a_{ss}+\lambda)e^{z_s}+\cdots+a_{sn}e^{z_n},$$

$s=1, \cdots, n$, the a_{sk} are real, continuous and bounded for $t\geqq t_0>0$ and λ is a real constant, suitably taken. The following theorem is stated. If $z_s=x_s+iy_s$ is a solution, with $0\leqq y_s\leqq\pi/4$ or $\pi\leqq y_s\leqq 5\pi/4$ for $t\geqq t_0>0$, then $\lim_{t\to\infty} z_s=-\infty$, $\lim_{t\to\infty} dz_s/dt=0$. [The proof given requires that, if $\pi\leqq y_s\leqq 5\pi/4$, λ must be sufficiently large and in the other range $(0, \pi/4)$, $-\lambda$ must be sufficiently large. In the theorem, $\lim z_s$ should read $\lim \Re z_s$.] *N. Levinson.*

Hole, Njål. Über eine Potentialfunktion, die eine exakte Integration der Schrödingergleichung gestattet. Norske Vid. Selsk. Forh., Trondhjem 13 (1940), no. 34, 139–142 (1941).

The equation $\psi''+(\epsilon-u)\psi=0$ can be integrated in terms of hypergeometric functions for $\epsilon<0$ if

$$u=c_1(q^2+1)^{-1}+c_2q(q^2+1)^{-2}+k^2(\tfrac{3}{4}q^2+\tfrac{1}{4})(q^2+1)^{-3},$$

$$kx=(q^2+1)^{\frac{1}{2}}+\tfrac{1}{2}\log\frac{(q^2+1)^{\frac{1}{2}}-1}{(q^2+1)^{\frac{1}{2}}+1}.$$

R. P. Boas, Jr. (Providence, R. I.).

Hole, N. Eigenwertbestimmung in der Wellenmechanik durch die Polynommethode. Norske Vid. Selsk. Forh., Trondhjem 14, no. 14, 51–54 (1941).

The author investigates solutions of the Schrödinger equation $\psi''+(\epsilon-u)\psi=0$, which can be written in the form $\psi=G(q)F(q)$, $q=q(x)$, where $F(q)$ satisfies the equation $qy''+(c-aq)y'-bay=0$. Using the condition that q is independent of ϵ, various representations are obtained for q. *R. Bellman* (Princeton, N. J.).

Furry, W. H. Two notes on phase-integral methods. Physical Rev. (2) 71, 360–371 (1947).

A re-derivation of the connection formulas for the asymptotic solutions of an equation $u''+k^2[y(x)+\lambda]u=0$ about a point at which $y(x)+\lambda$ has a zero of the first order. The method is that of Zwaan, but omits the assumption of reality of the equation's coefficients for real x. Some quantum mechanical applications, in which an outgoing wave for large positive x is called for, are discussed and a derivation of the normalizing constant for the approximate solution for the anharmonic oscillator is given. *R. E. Langer.*

Coronato, Savino. Teoremi di confronto per equazioni differenziali lineari del secondo ordine. Rend. Accad. Sci. Fis. Mat. Napoli (4) 9, 75–82 (1939).

For ordinary linear homogeneous differential equations of the second order the author establishes various comparison theorems by introducing suitable transformations which reduce the problem to the classical Sturmian case. In particular, by this method he proves criteria previously established by Fubini, Mammana and Cimmino. *W. T. Reid* (Evanston, Ill.).

Cambi, E. Una equazione differenziale del secondo ordine a coefficiente periodico reciproco di quello di Mathieu. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 1035–1041 (1946).

The author starts from the differential equation:

$$\frac{d^2y}{dx^2}+\frac{y}{r^2(1+2\gamma\cos x)}=0$$

which is of Hill's type if $2\gamma<1$. In this case a general Fourier series solution leads to the well-known Hill's determinantal equation from which the characteristic exponent may be derived. Approximate methods for its calculation, making use of continued fractions, are discussed. In the (r, γ)-plane regions of stability and instability exist, the boundaries of which coincide with parameter values for which periodic or half-periodic solutions of the differential equation are valid. These regions are discussed in some detail. *M. J. O. Strutt* (Eindhoven).

Cambi, E. Una equazione differenziale del secondo ordine a coefficiente periodico reciproco di quello di Mathieu. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 1181–1187 (1946).

Continuing the argument of the paper reviewed above, periodic solutions of the differential equation are discussed, including their orthogonal properties. Then generalizations of these solutions, valid for unrestricted parameter values, are given in the form of definite integrals in the complex plane. Their transformation properties, resembling the well-known hypergeometric theory, are discussed. Fourier transforms are used in obtaining further formulations of general integrals. No detailed method for the evaluation of characteristic exponents is given in the present general case of unrestricted parameters. *M. J. O. Strutt* (Eindhoven).

Cambi, Enzo. Una equazione differenziale del secondo ordine a coefficiente periodico reciproco di quello di Mathieu. Ricerca Sci. 17, 186–190 (1947).

Summary of the papers reviewed above.

Manacorda, T. Soluzioni periodiche di una equazione differenziale non lineare. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 1046–1050 (1946).

The author considers the equation $y''+\phi(x, y)=f(x)$, where $\phi(x, y)$ and $f(x)$ are continuous in the region $0\leqq x\leqq a$, $-\infty<y<+\infty$, and $\phi(x, y)$ satisfies the additional conditions $\phi(x, 0)=0$, $|\phi(x, y_1)-\phi(x, y_2)|\leqq b|y_1-y_2|$, b a positive constant, in this region. The following result is obtained. If $a<\pi b^{-\frac{1}{2}}$, there exists one and only one solution of the differential equation satisfying the conditions $y(0)=y(a)=0$. The method of proof is that of successive approximations, using a property of the Green's function for the equation $y''=0$ over the interval $0\leqq x\leqq a$. A consequence of the method is that, if $f(x)=0$, no solution not identically zero can satisfy the boundary conditions. The author uses this fact to obtain periodic solutions of the differential equation when $\phi(x, y)$ and $f(x)$ are periodic in x. *R. Bellman.*

***Borg, Göran. Inverse problems in the theory of characteristic values of differential systems.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 172–180. Jul. Gjellerups Forlag, Copenhagen, 1947.

The author surveys some results obtained earlier [Borg, Acta Math. 78, 1–96 (1945); these Rev. 7, 382], and indicates extensions to problems which are not self-adjoint. *H. Pollard* (Ithaca, N. Y.).

Wilkins, J. Ernest, Jr. The converse of a theorem of Tchaplygin on differential inequalities. Bull. Amer. Math. Soc. 53, 126–129 (1947).

The author considers the problem of deducing the inequality $v(x)>y(x)$, $x_0<x\leqq x_1$, from the relations

$$y''-p_1y'-p_2y-q=0,\qquad x\geqq x_0,\ y(x_0)=y_0,\ y'(x_0)=y_0',$$
$$v''-p_1v'-p_2v-q>0,\qquad x\geqq x_0,\ v(x_0)=y_0,\ v'(x_0)=v_0',$$

p_1, p_2, q continuous over the interval $x_0\leqq x\leqq x_1$. It is a question of determining the best possible value of x_1. It is shown that the value of x_1 depends upon nonvanishing properties of the solutions of the homogeneous equation $u''-p_1u'-p_2u=0$. *R. Bellman* (Princeton, N. J.).

Nasse, Gilbert. Sur les conditions de stabilité du circuit de régulation canonique d'une seule grandeur. C. R. Acad. Sci. Paris 224, 527–529 (1947).

By a canonical regulator circuit the author means one that contains a single regulated element and a single regulator element. Let $P_L=1/\varphi_\gamma$ and $P_D=\varphi_s/\varphi_e$ be the transmission ratios of the regulated and regulator element, respectively, the φ_γ, φ_s and φ_e being polynomials. Then, as is well known, the Laplace transform of $\gamma(t)$, the output of the regulated element, is of the form

$$T(\gamma)=\frac{\varphi_e I_L+I_D}{\varphi_e(P_D-1/P_L)}.$$

If $\varphi_\gamma\neq0$ and $\varphi_e\neq0$ in the right half-plane, the Nyquist criterion states that the circuit is stable provided that there is zero variation V in the argument of (P_D-1/P_L) on a Bromwich contour around the right half-plane. The author extends this criterion to the case that φ_e has, in the right half-plane, zeros of total multiplicity K_D. The circuit is stable in that case if $V=-2\pi K_D$. *M. Marden.*

Shtokalo, J. Linear differential equations of the n-th order with quasi-periodical coefficients. Rep. [Dopovidi] Acad. Sci. Ukrainian SSR no. 3–4, 17–20 (1946). (Ukrainian. Russian and English summaries)

Shtokalo, J. Systems of linear differential equations with quasi-periodical coefficients. Rep. [Dopovidi] Acad. Sci. Ukrainian SSR no. 3–4, 21–24 (1946). (Ukrainian. Russian and English summaries)

Shtokalo, J. Generalization of Heaviside's formula applied to linear differential equations with variable coefficients. Rep. [Dopovidi] Acad. Sci. Ukrainian SSR no. 3–4, 25–29 (1946). (Ukrainian. Russian and English summaries)

The results here appear to be those of Akad. Nauk. URSR. Informacilnil Byuleten no. 1(10–11), 38–39, 40–42, 42–45, 46–50 (1945); these Rev. 7, 520. A linear system with variable coefficients and right member of the form e^{pt}, where t is the independent variable, is considered. Conditions for the existence of a unique solution are given. The case of a general right member is treated by an appropriate generalization of the Bromwich integral of operational calculus. *N. Levinson* (Cambridge, Mass.).

Tzortzes, Anastasios. On the integration of partial differential equations by the method of infinitesimal transformations. Bull. Soc. Math. Grèce 21, 42–47 (1941). (Greek)

Rosenblatt, Alfred. On Cauchy's problem for a system of two partial differential equations of the first order with two unknown functions. II. Method of A. Haar. Actas Acad. Ci. Lima 9, 209–220 (1946). (Spanish)

[For part I see the same Actas 9, 139–152 (1946); these Rev. 8, 332.] Quelques fautes d'impression et quelque négligence dans la rédaction obligent le rapporteur à s'éloigner un peu de l'exposition de l'auteur. Le fait principal qui résulte du mémoire c'est [théorèmes 2 et 4] qu'étant donné un système de la forme

$$\partial z/\partial x=F(x, y, z, u, \partial z/\partial y),\quad \partial u/\partial x=\phi(x, y, z, u, \partial u/\partial y),$$

dont on cherche une solution telle que les fonctions inconnues z, u prennent des valeurs données sur un segment $-b\leqq y\leqq b$ de la droite $x=0$, si aux environs du dit segment et des valeurs données de z et u, les fonctions F, ϕ satisfont un système de conditions de la forme $(|\partial F/\partial z|, |\partial\phi/\partial u|)<B$, $(|\partial F/\partial z_y|, |\partial\phi/\partial u_y|)<A$, $(|\partial F/\partial u|, |\partial\phi/\partial z|)<Kf(x)$, où A, B, K sont des constantes et $f(x)$ une fonction donnée convenable, le dit problème de Cauchy ne peut avoir deux solutions distinctes; les théorèmes 2 et 4 correspondent respectivement à $f(x)=x^{\alpha-1}$ $(\alpha>0)$ et $f(x)=\alpha x^{-1}(\log x^{-1})^{-1-\alpha}$. *B. Levi* (Rosario).

Picone, M. Nuove vedute sull'integrazione delle equazioni lineari a derivate parziali. Ann. Sci. Univ. Jassy. Sect. I. 27, 18–26 (1941).

Le point de départ de l'auteur c'est ici, comme dans d'autres travaux, la considération que, si une fonction u appartient à une classe dans laquelle on connaît un système simplement infini de fonctions complet par rapport à l'intégration, cette fonction est déterminée et peut se calculer quand on connaît les valeurs de toutes les intégrales des produits de u par les fonctions du dit système. Il suppose alors donnés une équation linéaire aux dérivées partielles $E(u)=f$, d'ordre n avec r variables indépendentes, et un domaine D dont FD est la frontière; il appelle $E^*(v)$ l'expression différentielle adjointe de $E(u)$ et il observe qu'au

moyen de quelques intégrations par parties la formule de Green donne

$$(1)\quad \int_D uE(v)dX=\int_D fvdX+\sum_{k=1}^{n-1}\int_{FD}\frac{d^ku}{dl_k^k}E_k(v)d\sigma,$$

où les lettres l_k représentent $n-1$ directions fixées arbitrairement pour chaque point de FD, avec la seule condition de ne pas être tangentes à FD, et $E_k(v)$ sont des fonctions déterminées de v et de ses dérivées partielles. L'auteur indique par U_k les fonctions valeurs des d^ku/dl_k^k sur FD; si v est connue, (1) établit une relation qui permet déterminer la valeur de $\int_D uE^*(v)dX$, si les U_k sont connues. Si l'on connaît un système de fonctions v_s tel que le système $E^*(v_s)$ soit complet par rapport à une classe de fonctions à laquelle on sait que doit appartenir la fonction inconnue u, la connaissance des U_k permet de déterminer la fonction u. Il est évident qu'en supposant connues toutes les U_k on demande beaucoup plus que nécessaire. En effet la (1) se transforme en une relation entre les U_k seulement, si v est solution de $E^*(v)=0$. L'auteur amplie cette observation à divers cas hypothétiques, dans lesquels le nombre des U_k dont on demande la connaissance résulte réduit. Le mémoire, qui a plutôt la forme d'une notice que d'un travail détaillé, termine en posant une question sur le degré d'indétermination qu'il resulte pour les fonctions U_k. Le mémoire reproduit en partie un travail antérieur [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **75**, 413–426 (1940); ces Rev. **3**, 44].

B. Levi (Rosario).

***Cimmino, G. Nuove proprietà caratteristiche per le soluzioni delle equazioni lineari alle derivate parziali di tipo ellittico del secondo ordine.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 198–204. Edizioni Cremonense, Rome, 1942.

In a domain D, let

$$(1)\quad L(u)\equiv a_1u_{xx}+2a_2u_{xy}+a_3u_{yy}+a_4u_x+a_5u_y+a_6u=0$$

be a partial differential equation of the second order and of elliptic type. That is, the coefficients $a_j=a_j(x,y)$, which here are supposed to be functions having continuous partial derivatives of the first and second orders, satisfy $a_1>0$, $a_1a_3-a_2^2>0$. For a fixed point (x_0, y_0) in D, let $\rho^2=a_3(x-x_0)^2-2a_2(x-x_0)(y-y_0)+a_1(y-y_0)^2$, where a_1, a_2, a_3 are evaluated at (x_0, y_0), and let $E_0(r)$ denote the elliptical region $0\leqq\rho\leqq r$ in D. Then $u(x, y)$ satisfies a mean value property of the type

$$(2)\quad u(x_0,y_0)=\iint_{E_0(r)}K(x_0,y_0;x,y;r)u(x,y)dxdy.$$

Conversely, if $u(x, y)$ satisfies (2) for all $E_0(r)$ in D, then $u(x, y)$ satisfies (1). In what follows it is assumed that $a_6\equiv0$. Consequently, $u\equiv1$ is a solution of (1); and

$$\iint_{E_0(r)}K(x_0,y_0;x,y;r)dxdy=1.$$

It is shown that the kernel K can be chosen so as to be nonnegative for (x, y) in $E_0(r)$, at least for small values of r. The known result follows that, with $a_6\equiv0$, a nonconstant solution of (1) cannot have an interior maximum or minimum; consequently if, for each (x_0, y_0) of a Dirichlet domain relative to (1), there exists an $r_0=r_0(x_0, y_0)$ for which (2) holds, and $u(x, y)$ is continuous in the closure of the domain, then $u(x, y)$ satisfies (1).

E. F. Beckenbach.

Karimov, Dsh. H. Sur les solutions périodiques des équations différentielles non linéaires du type parabolique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 293–295 (1946).

For the nonlinear differential equation

$$z_t-a^2z_{xx}=\Phi(x,t)+\mu f(z),\qquad 0<t,\ x<1,$$

where $\Phi(x, t)=\Phi(x, t+1)=\sum_{n=1}^\infty\Phi_n(t)\sin n\pi x$, the following periodic boundary value problem is handled by the method of successive approximations: $z(0, t)=z(1, t)=0$, $z(x, 0)=z(x, 1)$. The existence of the solution is obtained for the parameter μ having any finite value. In an earlier paper μ was restricted [same C. R. (N.S.) **25**, 3–6 (1939); these Rev. **1**, 315].

F. G. Dressel (Durham, N. C.).

Cinquini-Cibrario, Maria. Intorno ad un sistema di equazioni alle derivate parziali del primo ordine. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **7(76)**, 177–184 (1943).

The Cauchy problem for a nonlinear hyperbolic partial differential equation (1) $F(x, y, z, p, q, r, s, t)=0$ has been reduced by H. Lewy [Math. Ann. **98**, 179–191 (1927)] to the problem of finding a solution of the canonical system

$$(2)\quad \sum_{k=1}^n a_{ik}(x,y,u_1,\cdots,u_n)\partial u_k/\partial x=c_i(x,y,u_1,\cdots,u_n),\qquad i=1,\cdots,m,$$

$$(3)\quad \sum_{k=1}^n a_{ik}(x,y,u_1,\cdots,u_n)\partial u_k/\partial y=c_i(x,y,u_1,\cdots,u_n),\qquad i=m+1,\cdots,n,$$

taking given values on $x+y=0$. In the original treatment of Lewy the latter problem is solved by the method of finite differences. The author points out that that problem can also be solved by differentiating (2) with respect to y, (3) with respect to x, and using iterations of the type of Picard on the resulting system. She finds, more generally, solutions of (2), (3) taking given values on a curve γ which is not parallel to the coordinate axes, or on two segments parallel to the axes [provided the data are compatible with (2), (3)]. [Reviewer's note: For application to (1), this method has already been used in Courant and Hilbert, Methoden der mathematischen Physik, vol. 2, pp. 326 ff.]

F. John (New York, N. Y.).

Cinquini-Cibrario, Maria. Sul problema misto per l'equazione del tipo iperbolico non lineare. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **7(76)**, 247–255 (1943).

Let $F(x, y, z, p, q, r, s, t)=0$ be a nonlinear second-order hyperbolic equation. Let γ_1, γ_2 be two arcs issuing from a point (x_1, y_1). In a previous paper [Ann. Mat. Pura Appl. (4) **21**, 189–229 (1941); these Rev. **6**, 4] the author showed the existence and uniqueness of a solution for prescribed values of u on γ_1 and γ_2, provided the characteristic directions at (x_1, y_1) (which depend on the data) do not separate the tangents of γ_1, γ_2 at that point. In the present paper the Cauchy data on γ_1 are prescribed and the values of u alone on γ_2. The author proves existence and uniqueness of a solution of this problem (in a suitable region) provided exactly one of the characteristic directions at (x_1, y_1) falls into the convex angle formed by γ_1 and γ_2. This result had been proved previously by Schauder [Studia Math. **6**, 190–198 (1936)]. The author here solves the problem by first finding a solution u_1 fitting the given Cauchy data on γ_1, the existence of which had been shown by H. Lewy. The solution u_1 exists up to a certain curve σ_2 which is characteristic with respect to u_1, passes through (x_1, y_1) and is tangent there to the characteristic direction which falls into

the convex angle between γ_1 and γ_2. The author then continues u_1 into the region between σ_2 and γ_2 by a solution u_2, which agrees with u_1 along σ_2 and takes the prescribed values on γ_2. The existence of u_2 follows from the author's previous result mentioned above. *F. John.*

Owens, O. G. Uniqueness of solutions of ultrahyperbolic partial differential equations. Amer. J. Math. **69**, 184–188 (1947).

The author proves three uniqueness theorems for solutions of the equation

$$\sum_{i=1}^{n} \partial^2 u/\partial x_i^2 - \sum_{k=1}^{m} \partial^2 u/\partial y_k^2 = 0$$

defined in a region V of $(x_1, \cdots, x_n, y_1, \cdots, y_m)$-space with boundary V^*. (I) If V is a sphere about the origin, then $u=0$ in V if $u=0$ on V^* and if the normal derivative of u vanishes in that portion of V for which $\sum_i x_i^2 - \sum_k y_k^2 \geqq 0$. (II) If V is a parallelepiped with faces parallel to the coordinate planes, then $u=0$ in V if $u=0$ on V^* and if the normal derivative of u vanishes on one face. (III) If V is bounded by hyperplanes $x_r=0$, $x_r+\sum_k b_k y_k=0$ (where $\sum_k b_k^2>1$) and by a cylinder with generators parallel to the x_r-axis, then u is uniquely determined by the values of u on V^* alone.

[These theorems illustrate the difficulty of generally devising appropriate boundary conditions for an ultrahyperbolic differential equation. It follows from results of the reviewer [Proc. Nat. Acad. Sci. U. S. A. **29**, 98–104 (1943); these Rev. **4**, 279] that theorems (I) and (II) cannot be the "best possible" ones in the sense that the normal derivative would only have to be prescribed on still smaller portions of V^*.] *F. John* (New York, N. Y.).

Mangeron, D. I. Sur les solutions par composition des équations aux dérivées partielles d'ordre supérieur. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] **1**, 295–303 (1946).

Some of the connections between solutions of elliptic, hyperbolic and parabolic differential equations which have been established by G. Doetsch [Math. Z. **46**, 315–328 (1940); these Rev. **1**, 314] and L. Koschmieder [Math. Z. **47**, 125–131 (1940); these Rev. **3**, 247] are extended here to solutions of corresponding differential equations of higher order. The author considers the generalizations $\partial^n u/\partial x^n \pm \partial^n u/\partial y^n = 0$, $\partial^n u/\partial x^n = \partial^{n-1} u/\partial x^{n-1}$ of the elliptic, hyperbolic and parabolic differential equations. As an example of the connections established here, if $u_h(x, y)$ and $u_p(x, y)$ satisfy the generalized hyperbolic and parabolic equations and certain conditions on the boundary of a rectangular region then the composition

$$U_{hp}(x, y) = \int_a^c u_h(t, y) u_p(t, x) dt$$

satisfies the generalized parabolic equation when n is an even integer. Tables are presented to show which type of equation is satisfied by each of the various compositions that can be formed. *R. V. Churchill.*

Mangeron, Dumitru Ion. Spectral problems and boundary problems for a class of linear differential equations with partial derivatives of higher order. Revista Ştiinţifică "V. Adamachi" **31**, 4 pp. (1945). (Romanian. French summary)

L'auteur étudie quelques problèmes à la frontière du domaine $a \leqq x \leqq c$, $b \leqq y \leqq d$ pour l'équation

$$\frac{\partial^{m+n} y}{\partial x^m \partial y^n} = \sum_{\mu=0}^{m-1} a_\mu \frac{\partial^{\mu+n} u}{\partial x^\mu \partial y^n} + \sum_{\nu=0}^{n-1} b_\nu \frac{\partial^{m+\nu} u}{\partial x^m \partial y^\nu} + \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{n-1} c_{\mu\nu} \frac{\partial^{\mu+\nu} u}{\partial x^\mu \partial y^\nu}.$$

Ces problèmes peuvent être appellés réductibles au sens de M. Picone, vu qu'ils se ramènent à des équations intégrales dépendantes de la construction des fonctions de Green relatives à des opérateurs d'un ordre inférieur et à un plus petit nombre de variables. Des exemples de noyaux caractéristiques pour quelques-uns de ces problèmes sont données. *From the author's summary.*

Diaz, Joaquin B. On a class of partial differential equations of even order. Amer. J. Math. **68**, 611–659 (1946).

Der Verfasser betrachtet partielle Differentialgleichungen der Form (1.1) $L^N u(x, y) = 0$, wo L ein linearer Differentialoperator zweiter Ordnung ist. Der Grundgedanke der Arbeit, welche diejenigen von Scheffers, Ward, Bers und Gelbart weiterführt, ist die Reduktion von (1.1) auf ein System (3.1) von $2N$ Differentialgleichungen erster Ordnung mit $2N$ Unbekannten mit Hilfe einer Algebra hyperkomplexer Zahlen. Eine hyperkomplexe Funktion aus dieser Algebra heisst regulär ("monogenic"), wenn ihre Koeffizienten das System (3.1) erfüllen. Für diese regulären Funktionen werden Differentiation und Integration definiert. Diese Operationen führen wieder auf reguläre Funktionen. Der Zusammenhang der regulären Funktionen mit (1.1) ist so, dass die Hälfte der (reellen) Koeffizienten einer regulären Funktion Integrale von (1.1) sind. Umgekehrt existiert zu jedem Integral von (1.1) eine reguläre Funktion, deren erste Komponente es ist. Da jede Konstante eine reguläre Funktion ist, kann aus ihr durch Iteration des Integrationsprozesses eine Folge $\{a \cdot Z^{(n)}\}$ von regulären Funktionen gebildet werden, die in Analogie zur klassischen Funktionentheorie die formalen Produkte von a mit den formalen Potenzen $Z^{(n)}$ heissen. Aus diesen Funktionen können formale Potenzreihen gebildet werden, für welche Konvergenzkriterien aufgestellt werden. Jede in einem Punkte reguläre Funktion kann in einer Umgebung desselben eindeutig durch eine formale Potenzreihe dargestellt werden. Aus den Komponenten der formalen Produkte der Basiseinheiten der Algebra mit allen formalen Potenzen kann ein vollständiges System reeller Integrale von (1.1) gewonnen werden. Während dies die wesentlichen Züge der Arbeit sind, wird durch einzelne Sätze der Zusammenhang mit der klassischen Funktionentheorie noch schärfer beleuchtet, so zum Beispiel durch die Verallgemeinerung des Cauchyschen Satzes und des Moreraschen. *W. Nef* (Fribourg).

Finzi, A. Sur les systèmes d'équations aux dérivées partielles, qui, comme les systèmes normaux, comportent autant d'équations que de fonctions inconnues. I, II. Nederl. Akad. Wetensch., Proc. **50**, 136–142, 143–150 = Indagationes Math. **9**, 99–105, 106–113 (1947).

Un système de n équations aux dérivées partielles $F_i=0$ d'ordre h, à n fonctions inconnues φ_j de $m+1$ variables indépendantes $x_0, \cdots, x_m$ est dit normal si, grâce éventuellement à un changement de variables, il peut être résolu par rapport aux dérivées $\partial^h \varphi_j/\partial x_0^h$; le théorème classique de Cauchy-Kowalevsky assure l'existence d'une solution et d'une seule d'un tel système lorsqu'on se donne sur une multiplicité (non caractéristique) à m dimensions les φ et leurs dérivées jusqu'à l'ordre $h-1$ (données analytiques bien entendu comme les équations). Un système anormal est caractérisé par le fait qu'une certaine combinaison des dérivées d'ordre $h(n-1)$ des équations données ne contient pas de dérivée d'ordre hn des φ. En joignant cette équation

à celles qu'on obtient en dérivant $hn-1-h$ fois $n-1$ des $F=0$ par rapport à une variable, on obtient un système d'ordre $h^{(1)}=hn-1$ qui peut être normal, système dont les solutions vérifient le système primitif pourvu seulement que les valeurs initiales des φ et de leurs $h^{(1)}-1$ premières dérivées satisfassent à certaines conditions. Janet avait donné [J. Math. Pures Appl. (9) **8**, 339–352 (1929)] ce théorème, en se limitant pour son exposé au cas $h=1$. Finzi n'a d'ailleurs eu connaissance de ce mémoire qu'après avoir obtenu ses résultats. Il applique le procédé précédent au cas où le système d'ordre $h^{(1)}$ obtenu est lui-même anormal; et, en supposant que par itérations successives on aboutisse à un système normal, il montre qu'on a encore une conclusion analogue.

L'introduction annonce que, dans une suite du mémoire, il sera démontré que, si au contraire la répétition du procédé ne conduit jamais à un système normal, le théorème d'existence prend un tout autre aspect: les F et leurs dérivées sont liées par une identité; on pourra encore assigner sur la multiplicité initiale les φ et leurs $h-1$ premières dérivées assujetties à des conditions convenables; mais la solution ne sera plus déterminée et on pourra lui imposer dans tout le domaine d'existence une condition arbitraire d'ordre h par rapport aux φ: le système donné pourra être comparé à un système de $n-1$ équations à n inconnues.

M. Janet (Paris).

Integral Equations

Srétensky, L. N. Sur la démonstration du théorème de Hilbert-Schmidt. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 194–197 (1946).

A well-known result in the Hilbert-Schmidt theory of integral equations is that, if $K(s, t)$ is a symmetric $\mathfrak{L}^2$ kernel, and $f(t)$ is an $\mathfrak{L}^2$ function orthogonal to all the characteristic functions $\varphi_i(t)$ of K, then $\int K(s, t)f(t)dt=0$. The author deduces this by a roundabout method from the Hilbert formula, which may be written in Hilbert space notation as

$$(Kf, g)=\sum_{i=1}^{\infty}(f, \varphi_i)(g, \varphi_i)/\lambda_i,$$

the λ_i being the characteristic values of K. He claims the merit of avoiding all mention of the iterated kernels K^n.

The result can be proved from the author's assumptions by the following argument. Let $(f, \varphi_i)=0$ (all i). Then $(Kf, g)=\sum_{i=1}^{\infty}\lambda_i^{-1}(f, \varphi_i)(g, \varphi_i)=0$ for all $\mathfrak{L}^2$ functions g. Take $g=Kf$, and we have $\|Kf\|^2=0$, $Kf=0$. *F. Smithies.*

Grioli, G. Relazioni quantitative per gli autovalori di un'equazione integrale omogenea di seconda specie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 522–526 (1946).

Let $K(s, t)$ be a positive definite symmetric kernel of integrable square whose lowest characteristic value λ_1 is of multiplicity 1, so that $0<\lambda_1<\lambda_2\leqq\lambda_3\leqq\cdots$, and suppose that $\tau(K)=\int K(s, s)ds=\sum\lambda_i^{-1}$ is finite. The author's results may be expressed in Hilbert space notation as follows: (i) $\lambda_1<\tau(K)/\|K\|^2$; (ii) $\lambda_1\leqq\lambda^*=(K\varphi, \varphi)/\|K\varphi\|^2$ for an arbitrary φ of integrable square; (iii) $\lambda_2>4\{\tau(K)\}^{-1}-\lambda^*$; (iv) if $\lambda^*\leqq 2/\tau(K)$, then $\lambda^*-\lambda_1<\lambda_2-\lambda^*$, i.e., λ^* is a closer approximation to λ_1 than to any other characteristic value; (v) if $\lambda^*\leqq 1.805/\tau(K)$, then $\lambda_2>2\tau(K)\cdot\|K\|^{-2}-\lambda^*$. A numerical illustration is given. *F. Smithies* (Cambridge, England).

Ascoli, G. Nuclei isotropi e loro autofunzioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 1167–1172 (1946).

The author considers a nucleus $K(P, Q)$ which is a function of points P, Q on an n-dimensional hypersphere and is invariant for any rotation of the hypersphere around its center. He shows that the characteristic functions of K are the n-dimensional hyperspherical functions and that conversely any continuous nucleus which has the n-dimensional hyperspherical functions as its characteristic functions is invariant in the above sense. *I. Opatowski.*

Sarmanov, O. Sur les solutions monotones des équations intégrales de corrélation. C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 773–776 (1946).

The author determines conditions under which there is a monotonic solution $\phi(y)$ of the equation

$$\lambda\int_a^b \phi(y)\frac{F(x, y)}{p(x)}dy=\phi(x), \qquad b-a=\infty,$$

where $F(x, y)$ is a symmetrical distribution function for the variables (x, y) and $p(x)$ is the prior distribution function of x. The main conclusion is that such a solution, which is unique, exists provided that the function

$$\Psi(x, y)=\int_0^y \frac{\partial}{\partial x}\frac{F(x, t)}{p(x)}dt$$

is of constant sign for $a\leqq x\leqq b$, $a\leqq y\leqq b$ and there exists a monotonic function $r(x)$ for which the successive functions $r_k(x)$ defined by

$$r_k(x)=\int_a^b r_{k-1}(y)\frac{F(x, y)}{p(x)}dy, \qquad r_0(x)=r(x),$$

satisfy $r_k(a)\Psi(x, a)=r_k(b)\Psi(x, b)=0$, $k=0, 1, \cdots$.

H. R. Pitt (Belfast).

Parodi, Maurice. Sur les solutions d'équations intégrales dont les noyaux renferment des polynomes d'Hermite. C. R. Acad. Sci. Paris **224**, 780–782 (1947).

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. XIII. Astrophys. J. **105**, 151–163 (1947).

[Part XII appeared in the same J. **104**, 191–202 (1946); these Rev. **8**, 178. This part is a continuation of part XI, same J. **104**, 101–132 (1946); these Rev. **8**, 59.] The author considers the diffuse reflection of a parallel beam of partially plane-polarized radiation by a semi-infinite plane-parallel atmosphere. By methods similar to those used in part XI, he obtains an approximate solution of the problem. He then shows that the emergent intensity components are linearly related to the flux components by relations involving a matrix **S**, which he calls the scattering matrix, and that Helmholtz's reciprocity principle corresponds to a symmetry property possessed by **S**. *F. Smithies.*

Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere. XIV. Astrophys. J. **105**, 164–203 (1947).

In earlier papers of this series [cf. the preceding review] approximate solutions of various diffuse reflection problems have been obtained. In the process the integro-differential equations to which the equations of transfer are reduced are replaced by finite systems of linear equations, a Gauss

quadrature being substituted for the integration in each equation.

In the present paper further steps are taken towards finding exact solutions. The author begins by writing the equations of transfer in vector form and derives a nonlinear integral equation for the scattering matrix **S** introduced in part XIII; the symmetry property of **S** derived there can be deduced directly from this equation. It is then shown that the solution of this equation can be reduced to the solution of equations of the form

$$(1)\qquad H(\mu)=1+\mu H(\mu)\int_0^1 \frac{H(\mu')}{\mu+\mu'}\Psi(\mu')d\mu',$$

where $\Psi(\mu)$ is an even polynomial of degree $2m$ such that $\int_0^1\Psi(\mu)d\mu\leqq\frac{1}{2}$. The last is a necessary and sufficient condition for $H(\mu)$ to be real and corresponds to the physical statement that no more radiation can be emitted in a scattering process than was incident. Assuming that the solution of (1) exists and is unique, the author derives a number of relations between the moments of the function $H(\mu)\Psi(\mu)$. These are used to transform (1) into a form suitable for numerical solution by an iterative process. They are also used in the special case of an electron-scattering atmosphere to verify the exact laws of darkening for the emergent intensities in the two different states of polarization.

Equation (1) is then replaced by the approximation

$$(2)\qquad H(\mu)\doteq 1+\mu H(\mu)\sum_{j=1}^{n}\frac{a_jH(\mu_j)\Psi(\mu_j)}{\mu+\mu_j},$$

where μ_j $(j=\pm1,\cdots,\pm n)$ are the zeros of $P_{2n}(\mu)$, $2n>m$, and a_j $(=a_{-j})$ are the weights in the corresponding Gauss quadrature formula. Let k_α $(\alpha=1,\cdots,n)$ be the distinct nonnegative roots of the equation

$$2\sum_{j=1}^{n}a_j\Psi(\mu_j)/(1-k^2\mu_j^2)=1.$$

It is shown that (2) is then satisfied by the function

$$H(\mu)=(\mu_1\cdots\mu_n)^{-1}\prod_i(\mu+\mu_i)/\prod_\alpha(1+k_\alpha\mu),$$

which may therefore be taken as an approximation to the solution of (1); closer approximations can then be obtained by the iterative process mentioned earlier.

The paper concludes by showing that the problems of diffuse reflection in accordance with the phase function $\lambda(1+x\cos\theta)$, where $\lambda\leqq1$, $-1\leqq x\leqq1$, or in accordance with Rayleigh's phase function, can both be reduced to nonlinear integral equations of the type considered. In the latter case the exact law of darkening is obtained.

F. Smithies (Cambridge, England).

Marshak, R. E. Note on the spherical harmonic method as applied to the Milne problem for a sphere. Physical Rev. (2) 71, 443–446 (1947).

In this paper the equation of transfer

$$\mu\frac{\partial\psi(r,\mu)}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial\psi(r,\mu)}{\partial\mu}+\psi(r,\mu)-\frac{1}{2}\int_{-1}^{1}\psi(r,\mu)d\mu=0$$

is solved by expanding $\psi(r,\mu)$ as a series of spherical harmonics and breaking it after a finite number of terms. The particular boundary conditions for which explicit solutions (in the first, second and third approximations) are obtained are those for an infinite medium with a "black core" of a given radius a: i.e., at $r=a$, $\psi(a,\mu)=0$ for $1\geqq\mu>0$. [The method of solution adopted in this paper is equivalent to that used by S. Chandrasekhar, Astrophys. J. **101**, 95–107 (1945); these Rev. **6**, 244.] *S. Chandrasekhar.*

Functional Analysis

Katětov, Miroslav. Über normierte Vektorräume. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **44**, 594–598 (1943).

This paper consists of some extracts from what is apparently primarily an expository article on aspects of the theory of normed linear spaces. However, the author gives new proofs of some of the theorems; in particular, he makes use of the Hahn-Banach extension theorem to avoid the direct use of transfinite induction. Moreover, he presents some theorems which, while probably more or less known, have not to the reviewer's knowledge been explicitly stated before. For example, he shows that a normed linear space which admits a reflexive subspace with a reflexive quotient space must itself be a reflexive space. *G. W. Mackey.*

Pinsker, A. G. On separable K-spaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) **49**, 318–319 (1945).

Several results concerning the representation of separable K-spaces are stated without proof. The following theorem is typical. Every separable K-space X may be represented as a disjunctive sum $X=\sum_\xi S_\xi+\sum_\xi s_\xi$, where the power of the set of summands is less than the power of the continuum and where S_ξ is contained in the space of measurable functions on (0, 1) and s_ξ is contained in the space of numerical sequences. *N. Dunford* (New Haven, Conn.).

Vulich, B. Sur les fonctionnelles linéaires dans les espaces semi-ordonnés linéaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 95–98 (1946).

Let X be a semi-ordered linear space satisfying the axioms (I)–(V) of Kantorovitch [Rec. Math. [Mat. Sbornik] N.S. **2(44)**, 121–165 (1937)]. Let X contain a unit 1, i.e., $\inf(1,x)>0$ for all $x>0$. An element e is called a quasi-unit in case $\inf(e,1-e)=0$. For every positive continuous linear functional F define e_0 as the supremum of all quasi-units on which F vanishes. Define $e(F)=1-e_0$ and $e^*=\sup e(F)$ as F varies over all positive continuous linear functionals. Let X^* consist of all elements x whose characteristic element $e_x\leqq e^*$. It is shown that the conjugate space $\bar{X}$ contains a unit if and only if there exists a linear functional on X essentially positive on all of X^*. A representation theorem for bounded linear functionals on X is also given for the case where $\bar{X}$ has a unit. *N. Dunford.*

Vulich, B. Sur les opérations linéaires multiplicatives. C. R. (Doklady) Acad. Sci. URSS (N.S.) **52**, 383–386 (1946).

An earlier result [same C. R. (N.S.) **41**, 142–144 (1943); these Rev. **6**, 130] is improved by proving that a continuous linear operation u between two partially ordered spaces X and Y, each with a unit, is multiplicative if and only if, for each quasi-unit e in X, $u(e)$ is a quasi-unit in Y. As a corollary it follows that (with a suitable choice of the unit in Y) a linear continuous operation u is multiplicative providing $u(x_1)$ and $u(x_2)$ are disjoint whenever x_1 and x_2 are disjoint. In the note mentioned above the author represented linear multiplicative operations by means of an abstract integral. In the present note he correlates the continuity properties of the operator with those of the measure function. *N. Dunford* (New Haven, Conn.).

Hoheisel, Guido. Existenz von Eigenwerten und Vollständigkeitskriterium. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1943**, no. 3, 7 pp. (1943).

Let L be a Hermitian (not necessarily bounded) operator in Hilbert space. The author shows that if L satisfies a certain set of conditions, chosen for their applicability to Sturm-Liouville operators, then its point spectrum is not empty; under some additional assumptions, the system of eigenfunctions of L is shown to be complete.

F. Smithies (Cambridge, England).

Castoldi, L. Operatori lineari nello spazio hilbertiano suscettibili di riduzione a forma normale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 141–146 (1947).

The author makes some elementary (and well-known) remarks concerning operators on Hilbert space and their expansions by means of series of proper vectors. A typical statement is that an operator [with pure point spectrum] is reducible [by a unitary transformation] to normal [diagonal] form if and only if its real and imaginary parts commute.

P. R. Halmos (Chicago, Ill.).

Powsner, A. On equations of the Sturm-Liouville type on a semi-axis. C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 295–298 (1946).

Consider the usual one-dimensional damped wave equation (E) with damping term $(\rho(x)-\rho(y))u$. The solution of the Cauchy problem $u=f(x)$, $u_y=0$ on $x=0$ is

$$f(x+y)-f(x-y)-2\int_0^\infty f(t)w(t,x,y)dt=2Tf.$$

[The author refers to another paper, unavailable to the reviewer, for the definition and facts connected with w, but it is evidently the Riemann function associated with (E) for $(x_0,t_0)=(t,0)$ so that the integral is actually over a finite range.] The main idea of the paper may be summed up by the remark that a ring multiplication is defined on an L_1 type of space generalizing the usual convolution definition, and involving the w kernel derived from (E). Thus let B be the Banach space of even functions in $L_1(-\infty,\infty)$ with $1+|x|$ as weight function. For each g in a certain subspace S of B an operator, roughly of the form $P_g(f)=\int_0^\infty(Tf)g dt$, is defined. The closure of finite sums of products of the P_g's determines a commutative ring which, with addition of a unity element and suitable norming, is a normed ring R in the Gelfand sense. Let Q refer to the functions in B vanishing outside a fixed interval. Then $Q\subset S$ and closure of Q in terms of the new norm $\|g\|=\|P_g\|$ leads to a ring R_1 isomorphic with R, where multiplication of f, g in R_1 is given by $P_f(g)$. Some applications are mentioned.

D. G. Bourgin (Urbana, Ill.).

Raikov, D. A. To the theory of normed rings with involution. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 387–390 (1946).

An operation $x\to x^*$ in a normed ring R is an "involution" provided (a) $(x+y)^*=x^*+y^*$, (b) $(\lambda x)^*=\bar\lambda x^*$, (c) $(x^*)^*=x$, (d) $(xy)^*=y^*x^*$. The set H of elements x such that $x^*=x$ is a real normed linear subspace of R. If $(x^*x+e)^{-1}$ exists for every x, then R is called "symmetric." A number of fundamental results of Gelfand and Neumark [Rec. Math. [Mat. Sbornik] N.S. **12**(54), 197–213 (1943); these Rev. **5**, 147] are extended to the case of a symmetric ring with a continuous involution. A linear functional f on H is said to be positive in case $f(x^*x)\geqq 0$. An involution is "essential" provided, for every x, there exists a positive f such that $f(x^*x)\neq 0$. An essential involution is continuous and, if R is symmetric, a continuous involution is essential. A ring with an essential involution is without radical. A special case here is the group ring of a locally compact group [Segal, Trans. Amer. Math. Soc. **61**, 69–105 (1947); these Rev. **8**, 438]. A necessary and sufficient condition for a ring with continuous involution to be symmetric is that $\sup f(x^*x)=\lim_{n\to\infty}\|(x^*x)^n\|^{1/n}$, where f is positive and $f(e)=1$. An example is given of a ring which is not symmetric but which possesses an essential involution.

C. E. Rickart (New Haven, Conn.).

Smith, R. C. T. The approximate solution of equations in infinitely many unknowns. Quart. J. Math., Oxford Ser. **18**, 25–52 (1947).

Consider the system (1) $x_\alpha-\lambda\sum_{\beta=1}^\infty k_{\alpha\beta}x_\beta=0$, $\alpha=1,2,\cdots$, where $\{k_{\alpha\beta}\}$ is the matrix of a completely continuous bilinear form in Hilbert space, for simplicity assumed to be Hermitian, and where, furthermore, $\sum_{\alpha=1}^m\sum_{\beta=m+1}^\infty|k_{\alpha\beta}|^2\to 0$ $(m\to\infty)$. Consider also the mth segment system (2) $x_\alpha-\lambda\sum_{\beta=1}^m k_{\alpha\beta}x_\beta=0$, $\alpha=1,2,\cdots,m$, with its m eigenvalues $\lambda_\alpha^{(m)}$, $\alpha=1,\cdots,m$. The problem is to examine how close the set $\{\lambda_\alpha^{(m)}\}$ comes to the set of the first m eigenvalues λ_α (in order of increasing magnitude) of system (1). Use is made of the known theory of the bilinear form $\{k_{\alpha\beta}\}$. By iteration, $x_{m+1}, x_{m+2},\cdots$ are expressed in terms of $x_1,\cdots,x_m$ by means of (1) for $\alpha=m+1, m+2,\cdots$ and the results are substituted into (1) for $\alpha=1,\cdots,m$. This reduces (1) to a finite system of form (3) $x_\alpha-\sum_{\beta=1}^m(\lambda k_{\alpha\beta}+r_{\alpha\beta})x_\beta=0$, $\alpha=1,\cdots,m$, where each $r_{\alpha\beta}$ is a function of λ. Comparison can then be made with system (2); upper and lower bounds are found for $\lambda_\alpha/\lambda_\alpha^{(m)}-1$, $\alpha=1,\cdots,m$. The definitions of the quantities present in these bounds are too long to set down here, but the bounds are such that $\lambda_\alpha^{(m)}\to\lambda_\alpha$ $(m\to\infty;$ $\alpha=1,2,\cdots)$.

It is pointed out that the method applies also when (1) is replaced by (4) $x_\alpha+\sum_{\beta=1}^\infty(h_{\alpha\beta}-\lambda k_{\alpha\beta})x_\beta=0$, $\alpha=1,2,\cdots$, the matrices $\{h_{\alpha\beta}\}$, $\{k_{\alpha\beta}\}$ fulfilling the same conditions as did $\{k_{\alpha\beta}\}$ in (1). An application is then made to an example, of form (4), encountered in the theory of elasticity.

I. M. Sheffer (State College, Pa.).

Theory of Probability

***Castelnuovo, Guido. Calcolo delle Probabilità. Volume I. Fondamenti della Teoria. Applicazioni alla Statistica, alla Teoria degli Errori, alla Balistica ed alla Fisica.** 3d ed. Nicola Zanichelli, Bologna, 1947. xxvii+321 pp. 700 lire.

The first edition appeared in 1918 and was appreciated as a progressive book with serious mathematical tendencies. During the intervening years both the theory of probability and mathematical statistics have undergone a radical change. Some of the newer developments in the theory of probability are to find place in a second, more mathematical, volume. The progress in statistical methodology is not reflected in the present volume and not mentioned in the introduction. This first volume serves as an elementary introduction more or less according to traditional lines: it

barely touches the substance of the theory of probability and dwells on its classical trimmings. The theory of probability is represented by the elementary composition rules, the binomial distribution with the normal and Poisson approximations, the central limit theorem (without proof), the simplest case of the law of large numbers [Cantelli], and the multidimensional normal distribution. Stieltjes integrals and characteristic or generating functions are not mentioned.

More than half of the book is devoted to various applications. The statistical theory is of the pre-Pearson and pre-Fisher type when Bayes's rule was accepted and Lexis's dispersion theory the height of sophistication. There are chapters devoted to a "derivation" of the normal law, to the theory of errors, ballistics and the Maxwell distribution of velocities. The exposition is clear and concise and makes the reader understand the original merits of the book.

W. Feller (Ithaca, N. Y.).

***Borel, Émile, et Chéron, André. Théorie Mathématique du Bridge à la Portée de Tous. 134 Tableaux de Probabilités avec Leurs Modes d'Emploi. Formules Simples. Applications. Environ 4000 Probabilités.** Monographies des Probabilités, Fasc. V. Gauthier-Villars, Paris, 1940. xx+392 pp.

The first chapter of this book contains an elementary discussion of the theory of the shuffle, concluding that a very simple shuffle is sufficient to ensure the validity of a large class of probability tables, but that it is more difficult to break enough two card sequences so that a player with an excellent memory cannot benefit by remembering the sequence of play from an earlier hand. Having concluded that elementary probability can be applied, the authors build the remainder of the book around a collection of 134 tables of probabilities for the distribution of hand patterns, suit patterns, etc. The tables were computed by machine, without recourse to Stirling's formula or other approximations; the results are presented to four and five significant figures. The authors discuss the application of the tables, mainly by way of illustrative examples, and draw some general conclusions. The tables in the second chapter concern the a priori distribution probabilities after the deal and before the hands are picked up. Those in the third chapter concern the distribution probabilities for the remaining three hands knowing the cards in one hand, with some conclusions on the defensive value of honors and on the choice of the opening lead. The fourth chapter contains the distribution probabilities for the two remaining unknown hands after the dummy is exposed. The fifth chapter discusses, in illustrative situations, the choice of line of play or contract which is best in the sense of the scoring expectation. The book concludes with a series of nine notes supplementing particular points in the earlier discussion.

L. H. Loomis (Cambridge, Mass.).

Birnbaum, Z. W., Raymond, J., and Zuckerman, H. S. A generalization of Tshebyshev's inequality to two dimensions. Ann. Math. Statistics 18, 70–79 (1947).

Les auteurs démontrent qu'étant données deux variables aléatoires W et Z indépendantes et non négatives, telles que $\lambda=E[W]\leqq E[Z]=\mu$, on a quel que soit t: (1) $\Pr[W+Z\geqq t]\leqq M(t)$, où $M(t)=1$ si $t\leqq\lambda+\mu$, $M(t)=\mu/(t-\lambda)$ si $\lambda+\mu\leqq t\leqq\frac{1}{2}(\lambda+2\mu+(\lambda^2+\mu^2)^{\frac{1}{2}})$, $M(t)=(\lambda+\mu)t^{-1}-\lambda\mu t^{-2}$ si $\frac{1}{2}(\lambda+2\mu+(\gamma^2+4\mu^2)^{\frac{1}{2}})\leqq t$; et pour des nombres $\lambda>0$, $\mu>0$, $\lambda\leqq\mu$ et $t>0$ donnés quelconques, il existe deux variables aléatoires indépendantes non négatives W et Z telles que $E[W]=\lambda, \mu=E[Z]$ et que l'égalité ait lieu dans (1); en posant $W=X^2/s^2$, $Z=Y^2/u^2$, $t=1$, $\lambda=\sigma_X{}^2/s^2$, $\mu=\sigma_Y{}^2/u^2$, où X et Y sont deux variables aléatoires indépendantes de moyennes nulles et d'écarts moyens quadratiques respectifs σ_X et σ_Y, ils deduisent de (1) des limitations de $\Pr[X^2s^{-2}+Y^2u^{-2}\geqq 1]$ qui améliorent l'inégalité de Bienaymé-Tschebyscheff [qui donnerait $\Pr[X^2s^{-2}+Y^2u^{-2}\geqq 1]\leqq\sigma_X{}^2s^{-2}+\sigma_Y{}^2u^{-2}$] et sont même les meilleures possibles sans hypothèses supplémentaires. L'extension à plus de deux variables est difficile, mais leur méthode conduit cependant les auteurs à des limitations qui, sans être les meilleures possibles, sont plus strictes que celles fournies par l'inégalité de Bienaymé-Tschebyscheff.

R. Fortet (Caen).

Kozakiewicz, W. On the convergence of sequences of moment generating functions. Ann. Math. Statistics 18, 61–69 (1947).

Soit X une variable aléatoire à 1 dimension, $F(x)$ sa fonction de répartition (f. r.), c'est-à-dire $F(x)=\Pr[X\leqq x]$, $\varphi(t)$ sa fonction génératrice (f. g.), c'est-à-dire $\varphi(t)=E[e^{tX}]$; l'auteur étudie la continuité de la correspondance entre F et φ, problème déjà envisagé par Curtiss [mêmes Ann. 13, 430–433 (1942); ces Rev. 4, 163], dont l'auteur complète les résultats en établissant trois théorèmes dont voici le troisième. Soit $\{F_n(x)\}$ une suite de f. r. et $\{\varphi_n(t)\}$ le suite des f. g. correspondantes dont on suppose qu'elles existent pour $|t|<\alpha$; une condition nécessaire et suffisante pour que $\{\varphi_n(t)\}$ converge dans l'intervalle $|t|<\alpha$ est que: (a) $\lim_{x\to\infty} M(x)e^{|t|x}=0$ pour $|t|<\alpha$, avec

$$M(x)=\limsup_{n\to\infty}[F_n(-x)+1-F_n(x)];$$

(b) $F_n(x)$ tende vers une f. r. $F(x)$ en tout point de continuité de $F(x)$; alors, la f. g. $\varphi(t)$ de $F(x)$ existe pour $|t|<\alpha$ et $\lim_{n\to\infty}\varphi_n(t)=\varphi(t)$, uniformément en t dans tout intervalle interieur à $(-\alpha, \alpha)$.

Dans une deuxième partie, l'auteur établit deux théorèmes analogues aux précédents, pour le cas d'une variable aléatoire à plusieurs dimensions, en se limitant, pour simplifier l'exposé, à 2 dimensions; il introduit pour cela, considérant une suite de f. r. à 2 dimensions $\{F_n(x_1, x_2)\}$ et la suite $\varphi_n(t_1, t_2)$ des f. g. correspondantes, les f. r. marginales $F_n{}^i(x_i)$ $(i=1, 2)$ relatives à $F_n(x_1, x_2)$ et les quantités $M_i(x_i)=\limsup_{n\to\infty}[F_n{}^i(-x_i)+1-F_n{}^i(x_i)]$, $i=1, 2$.

R. Fortet (Caen).

Bottema, O., and van Veen, S. C. Calculation of probabilities in the game of billiards. II. Nieuw Arch. Wiskunde (2) 22, 123–158 (1946). (Dutch)

[The first part appeared in the same Arch. (2) 22, 15–33 (1943); these Rev. 7, 209]. The present installment is devoted to handy analytical approximations to the results of the first part and to numerical computations.

W. Feller (Ithaca, N. Y.).

Hsu, P. L., and Robbins, Herbert. Complete convergence and the law of large numbers. Proc. Nat. Acad. Sci. U. S. A. 33, 25–31 (1947).

Les auteurs disent qu'une suite de variables aléatoires $\{X_n\}$, en général dépendantes, tend vers 0 complètement si

$$\lim_{n\to+\infty}\sum_{i=n}^{\infty}\Pr\{|X_i|>\epsilon\}=0$$

pour tout $\epsilon>0$, ce qui implique, comme on sait, la con-

vergence presque-certaine. Si l'on appelle F-équivalentes deux suites de variables aléatoires $\{X_n\}$ et $\{Y_n\}$ telles que, pour tout n, X_n et Y_n ont la même fonction de répartition, une condition nécessaire et suffisante pour que la suite $\{X_n\}$ tende complètement vers 0 est que toute suite F-équivalent à la suite $\{X_n\}$ tende presque-certainement vers 0. Les auteurs démontrent que: soit $\{X_n\}$ une suite de variables aléatoires indépendantes de même fonction de répartition, avec $E[X_n]=0$, et soit $Y_n=(X_1+\cdots+X_n)/n$; (a) si $E[X_n^2]<+\infty$, la suite Y_n tend vers 0 complètement; (b) si $E[X_n^2]=+\infty$, et si pour un α, $0<\alpha<2$, $E[|X_n|^\alpha]<+\infty$, la convergence de Y_n vers 0 n'est pas complète. Ils en déduisent un énoncé généralisé de la loi forte des grands nombres [dans cet énoncé, leur théorème 3, lire: $X_r^{(n)}$ au lieu de $X_n^{(r)}$ (première ligne) et: . . . pour $r\neq s$, au lieu de . . . pour $m\neq n$ (dernière ligne)]. *R. Fortet* (Caen).

Chung, Kai-lai. Note on some strong laws of large numbers. Amer. J. Math. **69**, 189–192 (1947).

Soient $\{a_n\}$ une suite indéfinie de nombres positifs croissants tendant vers $+\infty$, $\{X_n\}$ une suite indéfinie de variables aléatoires indépendantes de fonctions de répartition $V_n(x)$, et $S_n=\sum_{\nu=1}^n X_\nu$; l'auteur démontre le théorème suivant. Soit $\psi(x)$ une fonction paire, positive, non-décroissante pour $x>0$, et telle que: ou bien (a) $\psi(x)/x\downarrow$, ou bien (b) $\psi(x)/x\uparrow$, $\psi(x)/x^2\downarrow$ et $E(X_n)=0$ pour tout n; soit $M_n(\psi)=E[\psi(x_n)]$; si la série (1) $\sum_n M_n\psi/\psi(a_n)$ converge, la probabilité que $\sum_n X_n/a_n$ converge est 1; inversement, étant donnés des nombres $M_n(\psi)$ tels que la série (1) converge, il existe une suite $\{X_n\}$ de variables aléatoires indépendantes possédant des fonctions de répartition $\{V_n(x)\}$ telles que $M_n(\psi)=E[\psi(X_n)]$ et que la probabilité que $\sum_n X_n/a_n$ diverge est 1. De ce théorème, qui est une extension d'un résultat de Feller relatif au cas où $V_k(x)$ est indépendant de n [Amer. J. Math. **68**, 257–262 (1946); ces Rev. **8**, 37], l'auteur déduit divers énoncés du type "loi forte des grands nombres," par exemple: si $E(X_n)=0$, et s'il existe un nombre p compris entre 1 et 2, un $\epsilon>0$, et une constante C (indépendante de n) tels que

$$E\{|X_n|^p|\log|X_n||^{1+\epsilon}\}\leqq C,$$

on a presque-surement $\lim_{n\to\infty} S_n/n^{1/p}=0$. *R. Fortet.*

Bernstein, S. Sur le théorème limite de la théorie des probabilités. Bull. [Izvestiya] Math. Mech. Inst. Univ. Tomsk 3, 174–190 (1946). (Russian. French summary)

The Lindeberg conditions for the central limit theorem for independent variables have been generalized to the case of arbitrary norming constants and infinite moments [Feller, Math. Z. **40**, 521–559 (1935)]. These conditions are also necessary (the sufficiency is obvious in view of P. Lévy's results). The author emphasizes that the sufficiency of Feller's conditions follows also from Liapunov's conditions by means of the now usual method of truncation. The possibility of an extension to infinite moments has been mentioned by the author [Math. Ann. **97**, 1–59 (1926), in particular, p. 12]. The author's sufficient conditions are essentially equivalent to Feller's. [The reviewer regrets having overlooked the author's remarks hidden in considerations of dependent variables.] The author proceeds to show that also the classical Liapunov condition involving moments of order $2+\delta$ is necessary and sufficient for the tendency to the normal distribution if it is required that also all moments of order $\alpha<2+\delta$ tend to the corresponding moments of the limiting distribution. *W. Feller.*

Wald, Abraham. Limit distribution of the maximum and minimum of successive cumulative sums of random variables. Bull. Amer. Math. Soc. **53**, 142–153 (1947).

Soit, pour tout entier $N>0$, $X_{N1}, \cdots, X_{NN}$ des variables aléatoires indépendantes de même loi, d'écart moyen quadratique égal à 1, et de valeur moyenne μ_N; on pose $S_{N,k}=X_{N1}+\cdots+X_{Nk}$. Deux cas sont considérés: (a) la suite $\{\mu_N N^{\frac{1}{2}}\}$ tend vers une valeur finie lorsque $N\to\infty$; soit

$$P_N(a)=\Pr\{\max(S_{N,1},\cdots,S_{N,N})<aN^{\frac{1}{2}}\},$$
$$P_N^*(a,b)=\Pr\{-bN^{\frac{1}{2}}<\min_i(S_{N,i})\leqq\max_i(S_{N,i})<aN^{\frac{1}{2}}\},$$

où a et b sont deux constantes non negatives; l'auteur montre que les limites $\lim_{N\to\infty} P_N(a)$ et $\lim_{N\to\infty} P_N^*(a,b)$ existent et sont indépendantes de la distribution des X_{Ni} [résultat déjà obtenu par Erdös et Kac dans le cas particulier où $\mu_N=0$ et $a=b$; cf. Erdös et Kac, Bull. Amer. Math. Soc. **52**, 292–302 (1946); Kac, Ann. Math. Statistics **16**, 62–67 (1945); ces Rev. **7**, 459; **6**, 233; Wald suit un raisonnement analogue à celui de ces auteurs et utilise certains resultats de sa theorie des "tests progressifs"] et il donne des expressions de ces limites; (b) $\lim_{N\to\infty}\mu_N N^{\frac{1}{2}}=\infty$; alors, en posant $Q_N(c)=\Pr\{\max_i(S_{N,i})<N\mu_N+cN^{\frac{1}{2}}\}$, l'auteur montre que $\lim_{N\to\infty}Q_N(c)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{c}e^{-\frac{1}{2}t^2}dt$ (c est un nombre réel fixe quelconque). *R. Fortet* (Caen).

Bartlett, M. S. The large-sample theory of sequential tests. Proc. Cambridge Philos. Soc. **42**, 239–244 (1946).

The problem of finding the distribution of the sample size in sequential analysis is equivalent to the problem of the distribution of the absorption time in one-dimensional random walk with a drift and in the presence of absorbing barriers (Brownian motion in a field of constant force and in the presence of absorbing barriers). The asymptotic distribution is obtained by the familiar method of solving an appropriate parabolic differential equation. The method of images is used. The fact that the discrete problem can, in the limit, be treated as a continuous one is used heuristically.

[For recent literature on this and closely related problems see Wald, Ann. Math. Statistics **15**, 283–296 (1944); **16**, 117–186 (1945); these Rev. **6**, 88; **7**, 131, and the paper reviewed above; Tweedie, Nature **155**, 453 (1945); Kac, Ann. Math. Statistics **16**, 62–67 (1945); these Rev. **6**, 233. These writers, including the reviewer, have overlooked the fact that the discrete problem was already solved by De Moivre. For a review of the subject containing many original results and historical references, see E. C. Fieller, Biometrika **22**, 377–404 (1931). For connections with the theory of Brownian motion see M. Smoluchowski, Physik. Z. **17**, 557–571, 585–599 (1916), and the recent review article by Chandrasekhar, Rev. Modern Physics **15**, 1–89 (1943); these Rev. **4**, 248.] *M. Kac* (Ithaca, N. Y.).

***Kendall, M. G. Contributions to the Study of Oscillatory Time-Series.** National Institute of Economic and Social Research. Occasional Papers. IX. Cambridge, at the University Press; New York, The Macmillan Company, 1946. viii+76 pp. $1.75.

The author discusses stationary time series, that is, stationary stochastic processes, with particular reference to their periodicity properties. He is particularly interested in those whose chance variables are solutions of linear second order difference equations (*) $u_{t+2}+au_{t+1}+bu_t=\epsilon_{t+2}$, $t=0$, $\pm1,\cdots$, where a and b are constants and ϵ_{t+2} is independent of u_s for $s<t+2$. Since he neither formulates his problems in probability language nor uses the standard harmonic

analysis of stationary processes, he is reduced to making statements supported principally by the analysis of sample series obtained from (*) by choosing values of the constants a and b, obtaining ϵ_i's from sets of "random numbers." There is considerable discussion of periods and their meaning, but the author is apparently unaware of the significance of the spectral function of a stationary process and, in particular, of its role in defining the periods of such a process.

J. L. Doob (Urbana, Ill.).

Doob, J. L. Probability in function space. Bull. Amer. Math. Soc. **53**, 15–30 (1947).

Exposé clair et précis des fondements de la théorie des processus stochastiques dépendant d'un paramètre continu. Mesure de Kolmogoroff, difficultés. Conditions de Khintchine. Relativisation de la mesure, extension du domaine de mesurabilité [Doob]. Idée de Kakutani. Cas des processus mesurables [Doob et Ambrose]. Difficultés à résoudre. Possibilité de la considération des courbes en bloc.

M. Loève (London).

Fréchet, Maurice. Nouvelles définitions de la valeur moyenne et des valeurs équiprobables d'un nombre aléatoire. Ann. Univ. Lyon. Sect. A. (3) **9**, 5–26 (1946).

The author discusses generalizations of the median and mean values of a chance variable suggested by the problem of generalization to non-numerically valued chance variables. (In the following $k=1$ or $k=2$.) He favors the following definitions. Let X be the given chance variable and define $X_n^{(b)}=X$ if $|b-X|\leqq n$, $X_n^{(b)}=b$ otherwise. Then $\gamma^{(k)}$ is a typical value of order k if

$$(1)\quad \lim_{n\to\infty}[E\{|X_n^{(b)}-a|^k\}-\min_a E\{|X_n^{(b)}-a|^k\}]=h_k(a)$$

exists, is not constant and is independent of b; (2) $h_k(a)$ takes on its minimum value at $\gamma^{(k)}$. It is shown that these generalizations of the median ($k=1$) and mean ($k=2$) are actually generalizations of the usual concepts.

J. L. Doob (Urbana, Ill.).

Fréchet, M. Sur quelques idées modernes dans la théorie des probabilités. Acta [Trudy] Univ. Asiae Mediae. Ser. V-a. Fasc. **32**, 8 pp. (1940). (French. Russian summary)

Schützenberger, Marcel-Paul. Remarques sur des relations d'ordre entre variables aléatoires indépendantes. C. R. Acad. Sci. Paris **224**, 878–880 (1947).

Several types of order postulates applicable to chance variables are defined in an abstract way. No specific probability application is made. *J. L. Doob* (Urbana, Ill.).

Le Cam. Un instrument d'étude des fonctions aléatoires: la fonctionnelle caractéristique. C. R. Acad. Sci. Paris **224**, 710–711 (1947).

The author considers a stochastic process with chance variables $\{x(t)\}$ depending on the continuous parameter t and defines the characteristic functional

$$\text{Expectation }\{e^{i\int s(t)x(t)dt}\},$$

depending on the function $s(t)$. He states that various quantities are easily obtained using this functional.

J. L. Doob (Urbana, Ill.).

★Elfving, G. On a class of elementary Markoff processes. C. R. Dixième Congrès Math. Scandinaves 1946, pp. 149–159. Jul. Gjellerups Forlag, Copenhagen, 1947.

The author studies time discrete Markov chains in which the associated random variable $\mathbf{n}(t)$ can only increase in steps of 1. After recalling properties of the Pólya model of urns he finds all chains which share with the Pólya process the property that the probability of any particular sequence $\mathbf{n}(1)\to\mathbf{n}(2)\to\cdots\to\mathbf{n}(k)$ depends only on $\mathbf{n}(k)$, but not on the intermediate steps. This class of processes is in many ways analogous to the compound Poisson process in the time-continuous case. The latter can also be obtained by a passage to the limit. *W. Feller* (Ithaca, N. Y.).

Montroll, Elliott W. On the theory of Markoff chains. Ann. Math. Statistics **18**, 18–36 (1947).

Étude de fonctions additives et multiplicatives définies sur de chaînes simples de Markoff; intéressants exemples de nature physique. L'auteur ne semble pas connaître les travaux relativement récents, tels que ceux de l'école japonaise et le fondamental travail de Doblin [thèse, Paris, 1938]. *M. Loève* (London).

Blanc-Lapierre, André, et Fortet, Robert. Sur une propriété fondamentale des fonctions de corrélation. C. R. Acad. Sci. Paris **224**, 786–788 (1947).

Proof of Khintchine's theorem [Math. Ann. **109**, 604–615 (1934)] that the correlation function of a stationary stochastic process is the cosine Stieltjes transform of a monotone bounded function. The proof uses the theory of probability more explicitly than the usual one which bases the proof on the fact that the correlation function is positive definite. *J. L. Doob* (Urbana, Ill.).

Blanc-Lapierre, André, et Lapostolle, Pierre. Propagation d'une perturbation à spectre peu étendu dans un milieu dispersif non absorbant. Revue Sci. **84**, 579–595 (1946).

The author studies the propagation of a stationary stochastic process in a dispersive medium, concentrating on processes whose spectral distributions are confined to narrow bands and which therefore can be said to have carrier waves. Such processes have approximately sinusoidal sample functions; the author is particularly interested in the limits over which the obvious approximations are valid. The (one-dimensional) medium is supposed nonabsorbing and homogeneous; these hypotheses are interpreted to mean that the complex gain defining the transformation from one point to another h units away and s units later has the form $e^{2\pi i(\nu s-a(\nu)h)}$, where ν is the frequency and $a(\nu)$ is real.

J. L. Doob (Urbana, Ill.).

Cernuschi, Félix, and Castagnetto, Louis. Probability schemes with contagion in space and time. Ann. Math. Statistics **18**, 122–127 (1947).

A set of N urns is so arranged that each urn has m neighbors. Initially all urns have the same population of black and white balls. Consider an urn U and suppose that at the nth trial the total numbers of white and black balls extracted from the neighbors are l and s, respectively. Suppose, moreover, that the probability at the nth trial of extracting a white ball from U is ρ_n. The scheme considered consists in putting $p_{n+1}=p_na^lb^s$ or $p_{n+1}=p_nc^ld^s$, according as the nth trial with U results in a white or black ball. Some simple remarks concerning the asymptotic behavior of p_n follow.

W. Feller (Ithaca, N. Y.).

Hole, N. On the distribution of counts in a counting apparatus. Ark. Mat. Astr. Fys. **33B**, no. 8, 8 pp. (1947).

A new derivation of the results of Levert and Scheen [Physica **10**, 225–238 (1943); these Rev. **6**, 160]. Cf. also the author's paper in the same Ark. **33A**, no. 11 (1946); these Rev. **8**, 282. *W. Feller* (Ithaca, N. Y.).

Mathematical Statistics

✶Kendall, Maurice G. The Advanced Theory of Statistics. Volume II. Charles Griffin & Company Limited, 42 Drury Lane, London, W.C.2, 1946. vii+521 pp. £ 2/10/-.

[Vol. I appeared in 1944; cf. these Rev. **6**, 89.] This is the second volume of a very ambitious work. It takes off from the foundations described in volume I and is devoted chiefly to questions of estimation and testing hypotheses. The major topics discussed are: the maximum likelihood and other methods of point estimation, the Neyman theory of confidence limits, the Fisher theory of fiducial probability, regression, analysis of variance, design of sampling inquiries, significance tests, multivariate analysis and time series. A bibliographical discussion follows each chapter and a long bibliography is given at the end of the book. Each chapter is followed by exercises for the student, many of which are excellent. In fact, they supplement the text by setting as exercises for the student the derivation of results in cited papers. This is a valuable book. Not the least among its merits is the scope and volume of its contents, which are such that an advanced worker who wishes to read in a new (to him) branch of statistics will usually find sufficient preliminary orientation and an initial bibliography.

The extreme paucity of good books on mathematical statistics is such that a serious work with many merits such as this one, which represents the fruit of so much labor, deserves much commendation. Nevertheless certain serious adverse criticisms must be made.

(a) The first and most serious is that the level of mathematical rigor is not always satisfactory. This has two causes. Much of the now classical theory, which is British in origin, and even some of the contemporary British work, is written in a heuristic, nonrigorous fashion by people who use mathematics as many engineers and physicists do. To see what is meant consider, for example, the question of the number of degrees of freedom for the χ^2 test of goodness of fit when several parameters are estimated from the data. [The correct solution of this problem is, of course, due to R. A. Fisher.] Section 21.20 of the present book contains an argument of the type which for many years has passed as a proof of the correct result. This should be compared with a sound proof of the same result in Cramér's "Mathematical Methods of Statistics" [Princeton University Press, 1946; these Rev. **8**, 39]. Even if the nonrigorous argument had pedagogical value it would still be out of place in a book called "The Advanced Theory of Statistics." Actually it is the precise argument which is unambiguous and easier to understand, and the fuzzy argument which is so often incomprehensible. It is the necessity of putting these fuzzy arguments on a sound basis that makes the writing of good statistical books so difficult. Kendall has not, in this book, done much in this direction. What is worse, he has introduced errors of his own into results already on a sound basis. A few examples follow. (1) The first major proof with which the book begins, that of the optimum properties of the maximum likelihood estimate, is not, in the opinion of the reviewer, completely rigorous. The bibliographical appendix to the first chapter cites, however, a reference to a correct proof of this result by Doob. (2) Throughout the book the "delta method" (Taylor series) is used repeatedly, without validation. In this connection differentials are copiously abused. (3) The proof of the Fisher-Cochran theorem, fundamental in the analysis of variance, cannot be considered satisfactory as it is described in chapter 23. The correct proof is not difficult. The following questionable procedures advocated by many statistics books occur in various places in Kendall's book. (I) The pooling of estimates not found to be significantly different, even if there is no a priori reason for supposing equal the quantities being estimated. (II) The testing of hypotheses on the data which suggested them. (III) Repeated tests of significance on the same data.

(b) The Fisher theory of fiducial probability and the Neyman theory of confidence limits are treated "impartially," a separate chapter being given to each. The author notes that there are problems for which the two theories give incompatible results, but feels that he is keeping the treatment "objective." This reviewer is aware of the author's dilemma, but cannot regard the solution adopted as praiseworthy.

If the description of adverse criticisms appears longer than that of the favorable comments, it is only because one usually has to document the former rather more than the latter. The book is indispensable for advanced workers in the field, who are competent to be critical of occasional doubtful arguments, and capable of supplying their own where necessary. It is not, in the reviewer's opinion, very suitable for use as a textbook or by not very advanced students. *J. Wolfowitz* (New York, N. Y.).

Lovera, G. Metodo abbreviato di calcolo delle caratteristiche di una correlazione multipla. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **80**, 194–198 (1945).

Abbreviated calculation for the author's method [same Atti **77**, 341–346 (1942); these Rev. **7**, 463] for calculating multiple correlations and the like, based on the assumption of symmetrical relations between the variables.
J. W. Tukey (Princeton, N. J.).

✶Bonferroni, Carlo. Un indice quadratico di concentrazione. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 700–706. Edizioni Cremonense, Rome, 1942.

The author notes that the qualitative concept, concentration, widely used in statistics, covers a variety of associated but noncoincident concepts and cannot provide either a nominal definition or an explicit algorithm. Beside the known ratio of concentration R (which the author notes measures the disagreement between the curve of concentration of Lorenz and the corresponding line of equipartition) and the index B, introduced previously by the author [Elementi di Statistica, 1930, pp. 84–85], the author here proposes a quadratic ratio,

$$R_2=\left\{\int_{x_0}^{x_1}(x-M)^2y(x)dx\right\}\Big/\left\{\int_{x_0}^{x_1}x^2y(x)dx\right\}$$
$$=1-\left\{\int_{x_0}^{x_1}xy(x)dx\right\}^2\Big/\left\{\int_{x_0}^{x_1}x^2y(x)dx\right\}.$$

A. A. Bennett (Providence, R. I.).

✱Bonferroni, Carlo. Di un coefficiente di correlazione simultanea. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 707–714. Edizioni Cremonense, Rome, 1942.

In a previous paper the author generalized the concept of coefficient of correlation by introducing a "coefficient of parametric correlation" fitted to measure the degree of agreement between a diagram (x_i, y_i) and a linear combination with constant parameters of certain $k+1$ fundamental (linearly independent) functions of x. Here he extends the ideas and, by use of matrices and determinants, constructs a corresponding coefficient of simultaneous correlation for several diagrams; this coefficient assumes the extreme values ± 1 when and only when all the diagrams have exactly the desired dependence upon the basic functions.

A. A. Bennett (Providence, R. I.).

✱Salvemini, Tommaso. Di uno scarto trigonometrico medio, nel caso delle serie cicliche. Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 657–671. Edizioni Cremonense, Rome, 1942.

The author discusses the distinction between linear and quadratic measures for variability and correlation and the reasons for usually preferring the latter type of measure. In contrast to Gini's disagreement function for change of phase ("modality") for cyclic data, which uses the signed difference of phase of least absolute value, the author proposes the analytic (but essentially linear, as contrasted to quadratic) measure $1-\cos(x-x_k)$, x, x_k being expressed in terms of cyclotomic angles. He examines several previously published tables; his formula is seen to yield a more satisfactory graduation than any of the other four examined.

A. A. Bennett (Providence, R. I.).

Gini, Corrado. Degli indici sintetici de correlazione e delle loro relazioni con l'indice interno di correlazione (intraclass correlation coefficient) e con gli indici di correlazione tra serie di gruppi. Metron 14, 241–261 (1941).

Gini, Corrado. Alle basi del metodo statistico. Il principio della compensazione degli errori accidentali e la legge dei grandi numeri. Metron 14, 173–240 (1941).

Welch, B. L. On the Studentization of several variances. Ann. Math. Statistics 18, 118–122 (1947).

The author proposes an alternative approach to the development he discussed previously [Biometrika 34, 28–35 (1947); these Rev. 8, 394] for eliminating several variances simultaneously from probability statements concerning the mean of a normally distributed variable. Let an observed quantity y be assumed to be normally distributed about a population mean η with variances $\sigma_y^2=\sum_{i=1}^k\lambda_i\sigma_i^2$, where the λ_i are known positive numbers and the σ_i^2 are unknown population variances. Furthermore, let the data provide estimates s_i^2 of σ_i^2, based on f_i degrees of freedom. If the f_i are sufficiently large, the ratio $v=(y-\eta)/(\sum\lambda_i s_i^2)^{\frac{1}{2}}$ can be taken to be normally distributed with zero mean and standard deviation unity. For the f_i not necessarily large we may find some function $x=g(s_1^2, \cdots, s_k^2, y-\eta)$ still so distributed. The author first inverts, to obtain $y-\eta=h(s_1^2, \cdots, s_k^2, x)$. By a succession of transformations, integrations and introductions of auxiliary operators, the author obtains a procedure for a series approximation to x. He plans to continue the study and to provide tables for facilitating numerical computation.

A. A. Bennett (Providence, R. I.).

Levinsky, V. On the frequency constants of the sum of the several populations. Acta [Trudy] Univ. Asiae Mediae. Ser. V-a. Fasc. 26, 17 pp. (1939). (Russian. English summary)

Several statistical populations are "mixed" or "summed," i.e., the distribution function of the sum is a weighted mean of the several distribution functions. The author derives moments and related functions of the sum in terms of the corresponding functions of the several distribution functions.

J. Wolfowitz (New York, N. Y.).

Wishart, John. The cumulants of the Z and of the logarithmic χ^2 and t distributions. Biometrika 34, 170–178 (1947).

Exact expressions, finite series (for use with small numbers of degrees of freedom) and asymptotic series for the cumulants (semi-invariants in Thiele's terminology) of (A) $\log \chi^2$, (B) $z=\frac{1}{2}\log F$, (C) $\log t$. Asymptotic series for the first four cumulants and the measures of skewness and kurtosis of $(2\chi^2)^{\frac{1}{2}}$. An approximation to $\Gamma(X+\frac{1}{2})/\Gamma(X)$ with an error term $O(n^{-5.5})$. Data on accuracy of asymptotic approximations for moderate n. Some of the results for (A) have also been obtained by Bartlett and Kendall [see the following review] and by Rao [unpublished Calcutta thesis].

J. W. Tukey (Princeton, N. J.).

Bartlett, M. S., and Kendall, D. G. The statistical analysis of variance-heterogeneity and the logarithmic transformation. Suppl. J. Roy. Statist. Soc. 8, 128–138 (1946).

The cumulants of $\log s^2$, where the underlying variable is normally distributed, are obtained [this has also been done by Wishart in the paper reviewed above]. The first two cumulants γ_1 and γ_2 are tabled for $n=1(1)20$, together with a measure of the efficiency of the average of $\log s^2$ as an estimate of σ^2. The test of homogeneity of variance obtained by treating $\log s^2$ as normally distributed is briefly compared with Bartlett's test. The equation $W=UV$, where W and V have chi-square distributions with λ and n degrees of freedom, $\lambda<n$, and U and V are independent, is solved by a U with an incomplete-beta distribution. The application of these results to the use of the analysis of variance on variability data is considered briefly.

J. W. Tukey.

Bartlett, M. S. The general canonical correlation distribution. Ann. Math. Statistics 18, 1–17 (1947).

In the theory of the relations between two random vector variables, the sample characteristics known as canonical correlations r_i [Hotelling, Biometrika 28, 321–377 (1936)] are the roots of a determinantal equation. The joint distribution of these quantities is known in the special but important case when the true roots or correlations ρ_i are all zero. The present paper covers the general case of nonzero roots. The method used for obtaining the general distribution from the particular case of zero roots is analogous to the one adopted by Fisher [Proc. Roy. Soc. London. Ser. A. 121, 654–673 (1928)] in his derivation of the sampling distribution of the multiple correlation coefficient. Methods are given for obtaining the coefficients in the generalized hypergeometric expansions which are connected with the solutions, and various formulae are listed in two appendices, but the expressions obtained are too complicated to be quoted here.

H. Cramér (Stockholm).

The results for the z-distribution had been obtained previously by Aroian [Ann. Math. Statistics 12, 429–448 (1941); these Rev. 3, 175]

Roy, S. N. Multivariate analysis of variance: the sampling distribution of the numerically largest of the p-statistics on the non-null hypothesis. Sankhyā 8, 15–52 (1946).

This paper deals with the general distribution of the roots of the determinantal equations relevant to multivariate analysis, under the non-null hypothesis (that the population roots are non-zero). Sums of p-fold integrals are reduced by recursion formulas to sums of products of one-fold integrals of hypergeometric form. The initial general distribution was obtained by the author in a previous paper [Sankhyā 6, 35–50 (1942); these Rev. 4, 106]. In papers by T. W. Anderson [Ann. Math. Statistics 17, 409–431 (1946); these Rev. 8, 394] and M. S. Bartlett [see the preceding review] Roy's general result of the 1942 paper has been challenged, except for the case of only one non-zero root. Although the reviewer had not checked the details, the implication seems to be that the results of the present paper are valid only for the special case mentioned. *G. W. Brown* (Ames, Iowa).

Roy, S. N. A note on multi-variate analysis of variance when the number of variates is greater than the number of linear hypotheses per character. Sankhyā 8, 53–66 (1946).

In earlier papers [Sankhyā 3, 341–396 (1939); 6, 15–34, 35–50 (1942); 7, 133–158 (1945); these Rev. 4, 106; 7, 317, and the paper reviewed above], the author treated multivariate analysis of variance and the associated distribution theory of the roots of determinantal equations, under the usual restriction that the number of degrees of freedom for populations is at least as great as the number of characters. In this paper he treats the corresponding problems when the rank situation is reversed. The changes necessary in the results of the earlier papers are pointed out; the general nature of the results is the same, except that the determinantal equations now have at most $l-1$ non-zero roots, where l is the number of samples. The distribution theory for the general case, under the non-null hypothesis, is subject to the doubt pointed out in the preceding review.
G. W. Brown (Ames, Iowa).

Sillitto, G. P. The distribution of Kendall's τ coefficient of rank correlation in rankings containing ties. Biometrika 34, 36–40 (1 plate) (1947).

Suppose $r_1, \cdots, r_n$ is a given permutation of the integers $1, \cdots, n$. Consider the set of all pairs of the r's. Let each pair in which the second r exceeds the first r be assigned the "score" $+1$; otherwise the "score" is -1. Let S be the algebraic sum of all $n(n-1)/2$ "scores." Kendall's coefficient of rank correlation is given by (1) $\tau=2S/\{n(n-1)\}$. Kendall has found [Biometrika 30, 81–93 (1938)] the moments and distributions (for special values of n) of τ over the set of all permutations. Kendall therefore considers the case of no ties and zero "true" rank correlation. The range of τ is from -1 to $+1$.

Sillitto considers Kendall's problem for the case in which ties occur. If the set of n ranks contains p_2 pairs of equal ranks, τ is defined by Sillitto as (2) $\tau=2S/\{n(n-1)-2p_2\}$. In the case of n ranks containing p_2 pairs of equal ranks, p_3 triplets of equal ranks, $\cdots$, p_r r-tuplets of equal ranks, τ is defined by

$$(3)\qquad \tau=2S/\{n(n-1)-2p_2-6p_3-\cdots-r(r-1)p_r\}.$$

This makes it possible for τ (in the case of the ties just assumed) to range from -1 to $+1$.

Sillitto shows the variance of τ as given by (3) (under the assumption of zero "true" rank correlation) to be

$$\frac{n(n-1)(2n+5)}{18}-p_2-\frac{11}{3}p_3-\cdots-\frac{3+8+\cdots+(r^2-1)}{3}p_r.$$

He has also worked out the distribution of τ for $r=3$ and for various combinations of values of r, p_2, p_3, ranging from $n=3$ to 10, $p_2=1$ to 4 and $p_3=1$ to 3. *S. S. Wilks.*

Kimball, Bradford F. Sufficient statistical estimation functions for the parameters of the distribution of maximum values. Ann. Math. Statistics 17, 299–309 (1946).

A set of functions $T_i(O_n; a_1, \cdots, a_r)$, $i=1, \cdots, r$, of a sample point O_n and the parameters a_i on which the distribution of O_n depends is said to be a set of statistical estimation functions if the a_i are functions of O_n and the T_i. If, moreover, the density function $P(O_n; a_i)=g(T_i, a_i)h(O_n)$, the T_i are said to be sufficient; if this representation is possible when g is permitted to depend on T_i for values of a_i other than those appearing elsewhere in the equation, the T_i are said to be functionally sufficient. If the distribution (expectation) of the T_i is independent of the a_i, the T_i are said to be stable (stable in mean). It is pointed out that in general a set of maximum likelihood functions is functionally sufficient and stable in mean. For the density function $f(x)=\alpha ze^{-z}$ where $z=e^{-\alpha(x-u)}$ two pairs of sufficient estimation functions are exhibited and are shown to have asymptotic bivariate normal distributions. One of these is stable; the other (maximum likelihood) is stable in mean. A numerical illustration is given. *D. Blackwell.*

Brookner, Ralph J. Choice of one among several statistical hypotheses. Ann. Math. Statistics 16, 221–242 (1945). [MF 13914]

Let x be a (multi-dimensional) chance variable with density function $f(x, \theta)$ and let $H_1, \cdots, H_s$ be a division of θ-space into mutually exclusive sets. The following principle is proposed: choose that hypothesis H_i for which $\text{l.u.b.}_\theta\, W_i(\theta)f(x, \theta)$ is minimized, where $W_i(\theta)$ is the loss in accepting H_i when θ is the true value. With suitable weight function this principle leads to the usual test of Student's hypothesis. With suitable weight function, the decision regions are obtained for the hypotheses

$$\{\alpha_{i_1}=\cdots=\alpha_{i_k}=0,\quad \alpha_j\neq 0 \text{ for } j\neq i_1, \cdots, i_k\},$$

where $\alpha_1, \cdots, \alpha_p$ are the unknown means of p variables with a joint normal distribution and known covariance matrix. *D. Blackwell* (Washington, D. C.).

Brodovitskij, K. Sur le problème de ressemblance dans la théorie des échantillons statistiques. Acta [Trudy] Univ. Asiae Mediae. Ser. V-a. Fasc. 20, 38 pp. (1939). (Russian. French summary)

After some general discussion of testing hypotheses the author discusses the problem of testing the hypothesis of the identity of two normal populations. He comes to the conclusion that the t-test is unsatisfactory because it is not consistent with respect to alternatives where the two populations have the same mean but differ in their variances. [This is his conclusion phrased in modern terminology. Actually it is doubtful whether the notion of power was explicitly known to the author.] Thereupon the author concludes that a satisfactory solution is possible only with

the use of an a priori distribution of the parameters. This distribution, he says, cannot be arbitrary. Moreover, the assumption that this distribution is uniform (often called Bayes's solution) sometimes leads to contradictions. The author asserts that the analytic form of the distribution of the chance variables involved (the distribution depends upon the unknown parameters), determines the a priori distribution of the parameters. In a footnote it is stated that a paper by R. A. Fisher [Ann. Eugenics **6**, 391–398 (1935)], which was brought to the author's attention after his own paper had been written, contains the author's fundamental ideas, but its brevity does not permit detailed comparison between the two papers. The author's a priori probability is identified with Fisher's fiducial probability.

The reviewer was unable to follow the argument and doubts its validity. It runs counter to modern statistical ideas. A crucial point in the mathematical development, both here and in similar arguments, occurs when the probability density of the parameters, which had hitherto been regarded as unknown constants, first makes its appearance. Here this happens in equation (27) where differentials of the parameters are first written. [One wonders whether any difficulty would be occasioned in such arguments if the unknown parameters were of such a character that they could take only integral values, say.] *J. Wolfowitz.*

Brodovitskij, K. Sur les conditions nécessaires et suffisantes pour que les probabilités a priori aient une raison d'être. Acta [Trudy] Univ. Asiae Mediae. Ser. V-a. Fasc. **19**, 8 pp. (1939). (Russian. French summary)

Under certain analytic restrictions on the distributions of the chance variables, among which are the existence of sufficient estimates for the unknown parameters, the author derives the a priori distribution function whose existence he claims to have proved in the paper reviewed above. An antinomy of Cantelli's is said to be resolved by the use of this result. *J. Wolfowitz* (New York, N. Y.).

Bose, C., and Gayen, A. K. Note on the expected discrepancy in the estimation (by double sampling) of a variate in terms of a concomitant variate when there exists a non-linear regression between the two variates. Sankhyā **8**, 73–74 (1946).

Rider, P. Certain moment functions for Fisher's K-statistics in samples from a finite population. Acta [Trudy] Univ. Asiae Mediae. Ser. V-a. Fasc. **30**, 13 pp. (1939). (English. Russian summary)

Sample moments or, alternatively, Fisher's K-statistics, which are polynomials in the sample moments, vary from sample to sample and their higher moments as well as their means are of interest. Isserlis [Proc. Roy. Soc. London. Ser. A. **132**, 586–604 (1931)] attacked the higher moments of the sample moments from a finite population. The resulting expressions are cumbersome. The author calculates the higher moments of Fisher's k-statistics of total weight not exceeding 6 [$E((k_1)^2k_3)$ has weight 5]. He proceeds by equating coefficients in generating functions. Apparently without knowledge of the present paper, Irwin and Kendall [Ann. Eugenics **12**, 138–142 (1944); these Rev. **6**, 162] developed a method for this calculation and worked out the cases $E(k_2k_r)$ and all cases of weight not exceeding 4 except $E(k_1k_3)$. *J. W. Tukey* (Princeton, N. J.).

Egudin, G. I. On an effective method of calculation of the mathematical expectations of central sample moments. C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 487–490 (1946).

Operational calculi are presented for the expectations of the first and second powers of sample moments about the sample mean in terms of population moments and cumulants. These involve, for example, an operator O, where $O(U)=U\lambda_1$, $O(\lambda_k^a)=a\lambda_k^{a-1}\lambda_{k+1}$ and where O behaves formally like a differential operator. It is stated that known formulas are obtained with less work and greater accuracy. The validity of the results follows if the distribution has all moments and the limiting formulas for the expectations of the square of the sample moments are shown to hold under this hypothesis. *J. W. Tukey* (Princeton, N. J.).

Welker, E. L. The distribution of the mean. Ann. Math. Statistics **18**, 111–117 (1947).

The complete range of Pearson curves is considered as populations to be sampled. The sampling distributions of the mean as approximated by the Pearson system are given. Using the α_3^2, δ chart of Craig for the population, an analogous $\bar{\alpha}_3^2$, $\bar{\delta}$ chart is derived for the distribution of the mean. The properties of this one-to-one transformation are used to discuss the approach to normality of the distribution of the mean. *J. Janko* (Prague).

Leipnik, R. B. Distribution of the serial correlation coefficient in a circularly correlated universe. Ann. Math. Statistics **18**, 80–87 (1947).

Consider a "circular" sequence of random variables $x_0, x_1, \cdots, x_T=x_0$ such that the differences $s_t=x_t-\rho x_{t-1}$ ($|\rho|<1$; $t=1, \cdots, T$) are independent and normal with zero means and equal variances. Denote the (likewise equal) standard deviations of the x_t by σ. The scheme thus defined constitutes a natural simplification of the similar noncircular time series model.

As estimates for σ^2 and ρ one may use p/T and $\hat{r}=\hat{q}/p$, where $p=x_1^2+\cdots+x_T^2$ and $\hat{q}=x_1x_2+\cdots+x_Tx_1$. The exact distribution of $\hat{r}$ has been derived by R. L. Anderson in the case $\rho=0$ and by Madow in the general case, the analytic expression being, however, complicated. On the other hand, Koopmans and Dixon have introduced a simple approximate distribution in the case $\rho=0$. [See the same Ann. **13**, 1–13 (1942); **16**, 308–310 (1945); **13**, 14–33 (1942); **15**, 119–144 (1944); these Rev. **4**, 22; **7**, 131; **4**, 22; **6**, 6.] The author follows up the approximation line of research. By smoothing the joint characteristic function of p and $\hat{q}$, he deduces approximate expressions for the frequency functions of p and $\hat{r}$ in the case of an arbitrary ρ. For $\hat{r}$ he obtains

$$\bar{R}_\rho(\hat{r})=\frac{\Gamma(\frac{1}{2}T+1)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}T+\frac{1}{2})}(1-\hat{r}^2)^{\frac{1}{2}(T-1)}(1+\rho^2-2\rho\hat{r})^{-\frac{1}{2}T}.$$

Graphs of this distribution are shown for $T=20$ and various values of ρ. Furthermore, general expressions for the (approximate) moments of p and $\hat{r}$ are given and the mean and variance of these variables explicitly evaluated. *G. Elfving* (Helsingfors).

Walsh, John E. Concerning the effect of intraclass correlation on certain significance tests. Ann. Math. Statistics **18**, 88–96 (1947).

Several commonly used confidence intervals and significance tests based on the quantities t, χ^2, F, derived under the assumption of randomness, are considered; it is shown

how they vary if the values are from correlated "samples." Investigation of the effect of intraclass correlation then follows. *J. Janko* (Prague).

Jurgensen, C. E. Table for determining phi coefficients. Psychometrika 12, 17–29 (1947).

The author tables the "phi coefficient," given by

$$\phi=\frac{p_2-p_1}{(p_2+p_1)(2-p_2-p_1)}$$

to 3 decimal places for $0\leqq p_1\leqq p_2\leqq 1$ at intervals of 0.01 in each variable. This can be directly applied to the calculation of χ^2 for a four-fold table with one pair of marginal totals equal. *J. W. Tukey* (Princeton, N. J.).

Nandi, H. K. On the average power of test criteria. Sankhyā 8, 67–72 (1946).

The author defines average power over a family of surfaces $\psi(\theta)=\lambda$ in the parameter space and derives a sufficient condition for a region w_0 to have maximum average power over any surface $\psi(\theta)=\lambda$. The condition is applied with a suitable choice of $\psi(\theta)$ to the chi-square test, the analysis of variance test and to several tests applicable in multivariate analysis. In the case of the analysis of variance ratio a new proof of a result of A. Wald [Ann. Math. Statistics 13, 434–439 (1942); these Rev. 5, 129] is given. Although this does not affect the applications made by the author, it seems to the reviewer that the definition of average power given in the paper will depend not only on λ but also on the choice of $d\lambda$. *H. B. Mann* (Columbus, Ohio).

Mosteller, Frederick. On some useful "inefficient" statistics. Ann. Math. Statistics 17, 377–408 (1946).

Let a random variable be continuously distributed. If a sample $X_1', \cdots, X_n'$ is drawn from this distribution and is rearranged and renumbered so that $X_1<\cdots<X_n$ then the X_n's are called order statistics. The author considers functions of order statistics, so-called systematic statistics, and works out simple methods for estimating parameters to be used with punched-card equipment, with particular attention to cases where, in practice, the collection of data is inexpensive compared with the cost of analysis. He gives rapid methods for estimating the mean and the standard deviation when large samples from normally distributed variables are at hand and for estimating the correlation coefficient whether the parameters of the normal bivariate distribution are known or not. Finally the author compares the efficiency of the methods with the efficiency of other simple estimates. It is the intention of the author to give "inefficient" tests of significance in a later paper. *K. R. Buch* (Copenhagen).

Girshick, M. A., Mosteller, Frederick, and Savage, L. J. Unbiased estimates for certain binomial sampling problems with applications. Ann. Math. Statistics 17, 13–23 (1946). [MF 15954]

This paper initiates the theory of estimation for sequential sampling by considering the estimation of the parameter p for a binomial distribution. Let R be a region consisting of points (x, y) with nonnegative integral coordinates together with the point $(0, 0)$. A path from a point (x_1, y_1) to a point (x_n, y_n) is a sequence of points (x_i, y_i), $i=1, \cdots, n$, such that $x_{i+1}=x_i$, $y_{i+1}=y_i+1$ or $x_{i+1}=x_i+1$, $y_{i+1}=y_i$; R may not include points which cannot be reached by paths in R from $(0, 0)$. Sequential sampling from a binomial population generates a path beginning at $(0, 0)$. The sampling ceases when a boundary point of R is reached. For every boundary point α let $k(\alpha)$ be the number of distinct paths in R to that point from $(0, 0)$; then the probability that a particular sample will terminate at α is $k(\alpha)p^{x_\alpha}(1-p)^{y_\alpha}$, where (x_α, y_α) are the coordinates of α. Let $k^*(\alpha)$ be the number of paths in R from $(0, 1)$ to α. The estimator $\hat{p}$ for p is defined to be $k^*(\alpha)/k(\alpha)$. If R contains a finite number of points, $\sum_\alpha k(\alpha)p^{x_\alpha}(1-p)^{y_\alpha}=1$; if this relation holds for an infinite region, the region is said to be closed. A region is said to be simple if it has the convexity property relative to lines of the form $x+y=n$.

The following theorems are proved. (1) A sufficient condition that R is closed is that $\liminf A(n)n^{-\frac{1}{2}}=0$ as $n\rightarrow\infty$, where $A(n)$ is the number of points in R with $x+y=n$. (2) If S is a region in a closed R, S is closed. (3) For any closed R, $\hat{p}$ is an unbiased estimator of p. (4) A necessary condition that $\hat{p}$ is the unique unbiased estimator for a closed R is that R is simple. (5) A necessary condition that $\hat{p}$ is a unique unbiased estimator for a closed R is that there is no closed R' whose boundary is a proper subset of the boundary of R. (6) A sufficient condition that a closed R has $\hat{p}$ as a unique unbiased estimator is that R is simple and that there exist g, h $(0<g, h\leqq 1)$ such that for all boundary points $|gx-hy|<M$. (7) A necessary and sufficient condition that $\hat{p}$ is the unique unbiased estimator for a closed finite R is that R is simple. *A. M. Mood.*

Wolfowitz, J. On sequential binomial estimation. Ann. Math. Statistics 17, 489–493 (1946).

The results of the paper reviewed above are extended in two ways. In (2) of the preceding review, the condition $\liminf A(n)n^{-\frac{1}{2}}=0$ is replaced by $\liminf A(n)n^{-\frac{1}{2}}<\infty$; in (7) the restriction that R is finite is removed. *A. M. Mood* (Ames, Iowa).

Wolfowitz, J. Consistency of sequential binomial estimates. Ann. Math. Statistics 18, 131–135 (1947).

Girshick, Mosteller and Savage [see the second preceding review] have given an unbiased estimator $p(g)$ for estimating the parameter p of a binomial distribution on the basis of a sequential test. The author shows that this estimator is consistent in the sense of the following theorem. Let $T_1, T_2, \cdots$ be a sequence of sequential binomial tests. For the ith test T_i let n_i be the smallest integer such that $P(n=n_i)\neq 0$. Let $n_i\rightarrow\infty$ as $i\rightarrow\infty$. Then $p(g)$ converges stochastically to p as $i\rightarrow\infty$. *A. M. Mood.*

Haldane, J. B. S. On a method of estimating frequencies. Biometrika 33, 222–225 (1945). [MF 16453]

Let successive observations be made on a binomial chance variable until m "successes" have occurred, where $p>0$ is the probability of a success. Let the chance variable n denote the number of observations up to and including the one which yields the mth success. Then $E\{(m-1)/(n-1)\}=p$ [see also the third preceding review]. A series expansion for the variance of $(m-1)/(n-1)$ is obtained. An approximate estimate of the latter is $m(n-m)/n^2(n-1)$. If the population is multinomial and sampling proceeds until m members of one class have been obtained, and if m' is the number of observations belonging to another class whose probability is p', then $E\{m'/(n-1)\}=p'$, and an approximate estimate of $\sigma^2(m'/(n-1))$ is $m'(n-m')n^{-2}$. *J. Wolfowitz* (New York, N. Y.).

Wald, A., and Wolfowitz, J. Tolerance limits for a normal distribution. Ann. Math. Statistics 17, 208–215 (1946).

Tolerance limits which include at least the proportion γ of a normal population with probability β are given approximately by $\bar{x}\pm rs(n/\chi^2_{n,\beta})^{\frac{1}{2}}$, where $\bar{x}$ is the sample mean, $s^2=\sum(x-\bar{x})^2/n$, $n+1$ is the sample size, $\chi^2_{n,\beta}$ is the β level of the chi-square distribution with n degrees of freedom and r is the root of

$$\int_{\nu-r}^{\nu+r} e^{-\frac{1}{2}t^2}dt=(2\pi)^{\frac{1}{2}}\gamma, \qquad \nu=(n+1)^{-\frac{1}{2}}.$$

The error of the approximation is of order $1/n^2$.

A. M. Mood (Ames, Iowa).

Blackwell, David. On an equation of Wald. Ann. Math. Statistics 17, 84–87 (1946). [MF 15964]

The relation due to Wald [same Ann. 16, 117–186 (1945); these Rev. 7, 131], $E(x_1+\cdots+x_n)=aE(n)$ (where $E(x_i)=a$ and n is a function of $x_1, \cdots, x_n$), is shown to hold under either of the following general conditions when $E(n)$ is bounded: (1) the x_i have identical distributions; (2) the x_i are uniformly bounded. *A. M. Mood* (Ames, Iowa).

Blackwell, David. Conditional expectation and unbiased sequential estimation. Ann. Math. Statistics 18, 105–110 (1947).

After developing some results on conditional expectation, the author sets up a general method for obtaining an unbiased sequential estimator for a parameter admitting a sufficient statistic. If t is any simple unbiased estimator for the parameter (say based on the first observation of a sequential set of observations), and if $u_n(x_1, \cdots, x_n)$ is a sufficient estimator for a sample of size n, the author's estimator is defined in terms of the conditional distribution of t given u and n. It has the property that it is a function of the sufficient statistic for the parameter with respect to the observations of the sequential test. *A. M. Mood.*

Radhakrishna Rao, C. Minimum variance and the estimation of several parameters. Proc. Cambridge Philos. Soc. 43, 280–283 (1947).

A set of sufficient statistics for a multiparameter family of distribution functions is said to be minimal if it contains no more statistics than any other set of sufficient statistics. Let $\theta_1, \cdots, \theta_k$ be the parameters and let $t_1, \cdots, t_r$ be $r\leqq k$ independent statistics with expectations $\psi_i(\theta_1, \cdots, \theta_k)$ and covariance matrix V. Then (A) there is a set of r functions of a minimal set of sufficient statistics with expectations $\psi_i(\theta_1, \cdots, \theta_k)$ and covariance matrix U, where $V-U$ is positive definite or semi-definite. (B) Let I be the matrix $(\partial^2\phi/\partial\theta_i\partial\theta_j)$, where ϕ is the logarithm of the likelihood function, and Δ the matrix $(\partial\psi_i/\partial\theta_j)$. Then $V-\Delta I^{-1}\Delta'$ is positive definite or semi-definite. These results give minimum-variance properties of sufficient statistics and set lower bounds to the variance of unbiased estimates. [Result (A) has also been obtained in the one-parameter case by D. Blackwell [see the preceding review]; (B) is given for the case $r=k$ by H. Cramér, Mathematical Methods of Statistics, Princeton University Press, 1946, p. 497; these Rev. 8, 39.] These results, with the aid of some limit theorems of Wald, are used to show that, if L is the covariance matrix of the maximum likelihood estimates of $\theta_1, \cdots, \theta_k$ and V is that of any other set of consistent estimates of $\theta_1, \cdots, \theta_k$, then $nV-nL$ is, in the limit, positive definite or semi-definite. [The statements of the theorems do not always include all the conditions of their validity.]

The author has asked that the following misprints be noted. On page 280, in the line giving the definitions of I_{ij} and S_{ij}, the two expected values should be preceded by minus signs, and $p(S|\theta)$ should be replaced by $P(T|\theta)$. In the statement of theorem 1, $\psi_i(\theta_1, \theta_2, \cdots, \theta_r)$ should be replaced by $\psi_i(\theta_1, \theta_2, \cdots, \theta_k)$; also, in conclusion (B), insert $(i=1, \cdots, r; j=1, \cdots, k)$ after $(\partial\psi_i/\partial\theta_j)$. On page 282, in the first line of the proof of theorem 2, after "likelihood," add "and the uniform consistency of $\hat{\theta}_1, \cdots, \hat{\theta}_k$."

K. J. Arrow (Chicago, Ill.).

Mood, Alexander M. On Hotelling's weighing problem. Ann. Math. Statistics 17, 432–446 (1946).

The problem is to find the best method of weighing p objects by N weighing operations. The use of either a chemical or a spring balance, each with or without a scale bias, is discussed. Among various results the following may be noted. If an $N\times N$ Hadamard matrix H_N exists [Bull. Sci. Math. (2) 17, 240–246 (1893)], an optimum design for the chemical balance problem, with or without bias, is obtained from the matrix. Existence of H_{N+1} leads to an optimum design for the unbiased spring balance with $N=p$. When $N>p$ for this case and p is odd, the best procedure is to weigh the objects $\frac{1}{2}(p+1)$ at a time. The best design for the biased spring balance is easily obtained from the best design for the biased chemical balance. Designs suitable for small p are presented for both types of balance.

W. G. Cochran (Raleigh, N. C.).

Mathematical Biology

✶Hogben, Lancelot. An Introduction to Mathematical Genetics. W. W. Norton & Company, Inc., New York, 1946. xii+260 pp. $5.00.

An elementary presentation of mathematical methods and genetic results, intended for geneticists without mathematical training. The first four chapters deal with the probabilities (called throughout "proportionate possibilities") arising in various genetic situations, with elementary finite difference equations, and with the binomial frequency distribution. The later chapters consider gene-distribution problems under random (called "non-assortative") mating, the effects of various types of selection, the effects of assortative mating and inbreeding and the effects of mutation and selection. The genetic problems are confined to those involving a single gene substitution, on the ground that "it is relatively simple to extend" the results to any number of substitutions. *C. P. Winsor* (Baltimore, Md.).

Malécot, Gustave. Le calcul des probabilités et les problèmes de l'hérédité. Ann. Univ. Lyon. Sect. A. (3) 2, 25–37 (1940).

The bulk of the paper is devoted to a historical survey of the work of Karl Pearson, R. A. Fisher and others, in which the notions of Galton's biometric school are reconciled with the Mendelian theory of inheritance. Continuously variable characters like the body length are explained as the cumulative effect of the random contributions of a great number of genes. The correlation coefficients for successive generations can then be computed. The author recalls his methods [C. R. Acad. Sci. Paris 206, 153–155; 404–406 (1938)] which permit a considerable simplification of the necessary computations. *W. Feller* (Ithaca, N. Y.).

Andreoli, Giulio. Elementi matematici di base per una teoria delle collettività secondo la genetica. Rend. Accad. Sci. Fis. Mat. Napoli (4) 12, 40–47 (1942).

A few very general remarks about Mendelism and possible types of partial dominance in the case of multiple aleles. No mathematical results are given. *W. Feller.*

von Schelling, Hermann. Gedanken zum Weber-Fechnerschen Gesetz. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1944**, no. 5, 12 pp. (1944).

Let there be N receptors in the body capable of reacting to a certain stimulus (e.g., cells in the eye reacting to light). Assume that an individual receptor reacts if it is hit by more than k particles per second, and that the sensation is proportional to the number of receptors actually reacting. The number of particles hitting an individual receptor is supposed to be binomially distributed with the parameter p proportional to the local density of the incoming stream. Let R be the expected number of receptors which would react to a spatially homogeneous stream of intensity i. It is supposed that because of local fluctuations in density an actual stream of intensity i excites a number x of receptors determined by the classical random placement of R balls in N cells. With these assumptions one can write down the intensity of sensation as a function of i. It turns out that the middle section of the curve describes the Weber-Fechner law with reasonable accuracy; according to that law the intensity of the sensation changes by increments proportional to di/i. *W. Feller* (Ithaca, N. Y.).

TOPOLOGY

Eckmann, B. Topology and algebra. Gaz. Mat., Lisboa **8**, no. 29, 1–5; no. 31, 8–11 (1947). (Portuguese)

Translation of a lecture published in Vierteljschr. Naturforsch. Ges. Zürich **89**, 25–34 (1944); these Rev. **6**, 181.

Erdös, P. Some remarks on the theory of graphs. Bull. Amer. Math. Soc. **53**, 292–294 (1947).

Let $A(n)$ denote the greatest integer such that, given any graph of n vertices, either it or its complementary graph contains a complete subgraph of $A(n)$ vertices; e.g., $A(5)=2$. The author proves that, for $n\geqq 3$, $\frac{1}{2}\log_2 n<A(n)<2\log_2 n$. He then considers a more general result based on a combinatorial theorem of F. P. Ramsey. Finally, he establishes the following property of a set of $(k-1)(l-1)+1$ different integers: either there exist k of them, no one dividing another, or there exist l of them such that, when arranged in order, each is a multiple of its predecessor.
H. S. M. Coxeter (Toronto, Ont.).

Gomes, A. Pereira. Sur l'axiome de semi-régularité. Portugaliae Math. **5**, 207–217 (1946).

The author introduces a new axiom for topological spaces. A topological space X is said to be semi-regular if, for every closed subset F of X such that $F'\neq 0$, there exists an open set G such that $G\neq 0$ and $G^{-}\subset F'$. Six equivalent forms of this axiom are listed and it is proved that the semi-regularity axiom is independent of various well-known axioms for topological spaces. The discussion is concerned with topologized Boolean algebras, the study of which may easily be reduced to the study of true topological spaces. It may be noted that the author's use of the term semi-regular does not coincide with M. H. Stone's [Trans. Amer. Math. Soc. **41**, 375–481 (1937), pp. 442 ff.]. *E. Hewitt* (Chicago, Ill.).

Sierpinski, Waclaw. Sur une propriété des espaces métriques dénombrables. Portugaliae Math. **5**, 193–194 (1946).

It is proved that of any two denumerable metric spaces at least one is a continuous image of the other. One observes at the start that each is a subset of the real line [cf. the author's "Introduction to General Topology," University of Toronto Press, 1934, p. 103]. *R. Arens.*

Wu, Wen-Tsün. Note sur les produits essentiels symétriques des espaces topologiques. C. R. Acad. Sci. Paris **224**, 1139–1141 (1947).

If E is a space, the essential symmetric product $E(n)$ is obtained from the Cartesian product E^n by identifying $(x_1,\cdots,x_n)$ and $(y_1,\cdots,y_n)$ if one set is a permutation of the other. The homology groups of $I(n)$ and $C(n)$ are established, where I is the 1-cell and C is the 1-sphere.
R. Arens (Los Angeles, Calif.).

de Groot, J. A note on 0-dimensional spaces. Nederl. Akad. Wetensch., Proc. **50**, 131–135 = Indagationes Math. **9**, 94–98 (1947).

The author announces (without proof) the following results. (a) By taking the class of open-closed subsets of a separable, n-dimensional ($n>0$) totally disconnected space M as a subbase, one obtains a topological space $Q(M)$ ("space of quasi-components") which has no countable base. (b) A separable space M has a compact separable space $Q(M)$ of quasi-components only if it is compact.
R. Arens (Los Angeles, Calif.).

Arens, Richard. Topologies for homeomorphism groups. Amer. J. Math. **68**, 593–610 (1946).

L'auteur considère deux topologies sur le groupe H des homéomorphismes d'un espace localement compact A: la k-topologie, qui n'est autre que la topologie de la convergence uniforme sur les parties compactes de A; et la g-topologie, qui est induite sur H par la k-topologie du groupe H^* des homéomorphismes de l'espace A^* obtenu en rendant A compact par adjonction d'un point à l'infini. Pour la k-topologie, le composé fg de deux homéomorphismes est fonction continue de (f, g), mais en général f^{-1} n'est pas fonction continue de f; il en est toutefois ainsi lorsque A est compact ou localement connexe, et dans ce cas H, muni de la k-topologie, devient un groupe topologique; pour la g-topologie, H est toujours un groupe topologique.

L'auteur montre que la g-topologie sur H est identique à celle qu'on déduit de la structure uniforme suivante: pour tout entourage U d'une structure uniforme sur A (compatible avec la topologie de A) et toute partie compacte K de A, on prend comme entourage l'ensemble des couples (u, v) d'homéomorphismes tels que pour tout $x\in K$, $(u(x), v(x))\in U$ et $(u^{-1}(x), v^{-1}(x))\in U$ [cf. J. Braconnier et J. Colmez, C. R. Acad. Sci. Paris **223**, 230–232 (1946); ces Rev. **8**, 49]. Il prouve aussi que le groupe H (muni de la g-topologie) est complet pour sa structure uniforme bilatère [pour la structure uniforme droite ou gauche, non seulement le groupe H n'est pas complet en général mais il ne peut même être complété; cf. J. Dieudonné, C. R. Acad. Sci. Paris **218**, 774–776 (1944); ces Rev. **7**, 241].

L'auteur considère ensuite les sous-groupes de H qui sont (uniformément) équicontinus; il prouve que si A est connexe, un tel sous-groupe est partout dense dans un sous-groupe localement compact de H (toujours pour la g-topologie). Il donne des conditions pour qu'un domaine de transitivité d'un groupe topologique d'homéomorphismes de A puisse être identifié (du point de vue topologique) à un espace homogène déduit de ce groupe. Enfin, il examine les relations entre la topologie usuelle d'un groupe de Lie (de transformations analytiques) et la g-topologie sur un tel groupe.

J. Dieudonné (São Paulo).

✱**Fenchel, W. On closed surfaces with constant sign of curvature in the projective space.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 207–212. Jul. Gjellerups Forlag, Copenhagen, 1947.

The following theorem is proved. Let f be an abstract closed surface and φ a model of it in the projective space. If the sign of the relative curvature of φ is positive, f is homeomorphic to the sphere. If it is negative, f is homeomorphic to the torus or to Klein's bottle. If the relative curvature is zero, f is homeomorphic to the sphere or the projective plane. Examples are given showing that the different types of surfaces exist.

S. Chern (Shanghai).

Ehresmann, Charles. Sur les sections d'un champ d'éléments de contact dans une variété différentiable. C. R. Acad. Sci. Paris **224**, 444–445 (1947).

Let Φ be a field of $(n-p)$-dimensional elements of contact in a differentiable manifold V_n. A cross section of Φ consists of a differentiable manifold V_p and a map f: $V_p \to V_n$, such that for an arbitrary $x \epsilon V_p$ the image under f of the tangent space to V_p at x is nondegenerate and intersects the element of Φ at $f(x)$ in the zero-vector only. Theorem: let (V_p, f) be a cross section of Φ with orientable compact V_p; if the image $f_* \Gamma_p$ under f of the basic p-cycle Γ_p of V_p is homologous to 0 in V_n (rational coefficients), then the characteristic $\chi(V_p)$ is 0. The proof utilizes a fiber bundle E, with base space V_n; the fiber over any $y \epsilon V_n$ is the tangent space at y, taken modulo the element of Φ at y, and with origin removed. Let W^p be the Stiefel-Whitney characteristic cocycle of V_n; then one has $\chi(V_p) = f^* W^p \cdot \Gamma_p = W^p \cdot f_* \Gamma_p = W^p \cdot 0 = 0$. A more general theorem concerning other characteristic classes is given; in particular, if $V_n = S_n$, the n-sphere, then all characteristic classes of V_p vanish.

H. Samelson.

Alexander, J. W. Gratings and homology theory. Bull. Amer. Math. Soc. **53**, 201–233 (1947).

This paper contains a new approach to homology theory, applicable to topological spaces and more general systems, like lattices. The theory is developed first for an abstract algebraic system, the grating, in which homology-rings are defined. Roughly speaking one deals with a collection of finite (cochain) complexes, and an identification procedure, corresponding to subdivision, somewhat as in Čech cohomology theory or in Leray's recent theory [J. Math. Pures Appl. (9) **24**, 95–167, 169–199 (1945); 201–248 (1946); these Rev. **7**, 468]. A cut is an ordered set of three elements a, b, c, called the negative face, the edge, the positive face of the cut. A grating is a collection of disjoint cuts. (For a space X there are several natural gratings; the cuts are (1) the real valued continuous functions γ, with a, b, c corresponding to the sets where $\gamma(x) < 0, =0, > 0$, (2) the closed sets, (3) ordered pairs (A, C) of disjoint open sets.) For each ordered set $\alpha = (\gamma_1, \cdots, \gamma_n)$ of n ($\geqq 0$) cuts one considers a complex of type α; its cells are the symbolic noncommutative products $z_1 \cdot \cdots \cdot z_n$, where each z_i is the a, b or c of γ_i; n is the degree; the number of z's which are b's is the rank of the cell. (For $n=0$ there is only one cell E_ϵ of unit type ϵ.) Chains K, L, etc., are linear combinations of the cells with coefficients from a ring. (There is a zero-chain 0_α for every type α.) Cells and, by distributivity, chains of different types can be multiplied in the obvious way. There are three operators which preserve the type: the rank operator $^\rho$ which annihilates all cells not of rank ρ, the operator * which changes the sign of all cells of odd rank, and the boundary operator $'$, defined by $E_\epsilon' = 0$, $a' = -b$, $b' = 0$, $c' = b$ for the a, b, c of a cut, and the formula $(KL)' = K'L + K^*L'$. The familiar formula $K'' = 0$ holds. Relations between chains of different types are established with the help of refinements. The refinor E_α of type α is the sum of all cells of type α and rank 0; the refinement of K by E_α is KE_α or $E_\alpha K$. Two chains K_α and L_β are pseudo-added (and similarly pseudo-subtracted) by the formula $K_\alpha \oplus L_\beta = K_\alpha E_\beta + E_\alpha L_\beta$; this chain has type $\alpha\beta$. An ideal of a grating Γ is a subset which contains 0_ϵ, which contains with two chains of equal type their sum and which satisfies a third condition [see below]. A chain K is a cycle mod an ideal Φ if $K' \epsilon \Phi$. A chain K bounds in an ideal Ψ, mod an ideal Φ, if $K = W' + Z$ with $W \epsilon \Psi$ and $Z \epsilon \Phi$; notation: $K \approx 0(\Psi/\Phi)$. Two chains K_α and L_β are homologous, if there exists a refinement of their pseudo-difference which bounds in Ψ mod Φ, i.e., if $(K_\alpha E_\beta - E_\alpha L_\beta)E_\gamma \approx 0(\Psi/\Phi)$ for suitable E_γ.

Homology divides the cycles of Ψ mod Φ into disjoint homology classes. One proves that pseudo-addition and multiplication of representatives turns the set of homology classes into a ring, the homology ring $\mathcal{H}(\Psi/\Phi)$. The rank operator carries over to this ring; the additive group of the ring is the direct sum of the groups $\mathcal{H}^\rho$ of the various ranks ρ; the products obey the usual anti-commutative law of cohomology theory, with rank replacing the dimension. If Φ is the ideal consisting of all the zeros 0_α, then $\mathcal{H}(\Psi/\Phi) = \mathcal{H}(\Psi)$ is called the ring of Ψ. It turns out that $\mathcal{H}(\Gamma) = \mathcal{H}^0(\Gamma) =$ ring of coefficients, so that a significant theory is obtained only by restriction to properly chosen ideals.

A representation of a grating Γ on a space X is a function f from $\Gamma \times X$ to $(-1, 0, 1)$; every cut $\gamma = (a, b, c)$ determines three subsets A, B, C of X, the sets where $f(\gamma, x) = -1, 0, 1$, respectively. The loci of a, b, c are $A \cup B$, B, $B \cup C$; the loci of chains are defined in a straightforward manner. Every grating Γ defines its intrinsic carrier X_Γ, the space whose points are the functions from Γ to $(-1, 0, 1)$; there is a natural intrinsic representation of Γ on X_Γ, which defines the notion of intrinsic locus $\|K\|$ of a chain K. Condition 3 for ideals [see above] says that if $K \epsilon \Phi$ and $\|L\| \subset \|K\|$ then $L \epsilon \Phi$.

A representation of Γ on X determines two ideals of Γ: the ideal Φ_0 of chains with empty loci and the ideal Ψ_c of chains with compact loci. A significant homology theory is obtained by considering the ring $\mathcal{H}(\Psi_c/\Phi_0)$. [The use of Φ_0 here seems to correspond to a similar operation in Leray's theory.] The natural gratings of a space X, mentioned at the beginning, have natural representations on X; for (1) the representation is given by $f(\gamma, x) = \operatorname{sign} \gamma(x)$; for (2) $f(\gamma, x)$ is $-1, 0, 1$ if x belongs to the interior, frontier or exterior of γ; for (3) f is obvious. For each grating one obtains in $\mathcal{H}(\Psi_c/\Phi_0)$ a topologically invariant homology ring of X. In another paper, relations between these rings will be discussed and applications will be given.

H. Samelson (Ann Arbor, Mich.).

Hu, Sze-tsen. Concerning the homotopy groups of the components of the mapping space Y^{S^p}. Nederl. Akad. Wetensch., Proc. **49**, 1025–1031 = Indagationes Math. **8**, 623–629 (1946).

Let Y be a compact absolute neighborhood retract and let G^p be the function space of maps of a p-sphere into Y. Recently G. W. Whitehead obtained several connections between the homotopy groups of G^p and those of Y [Ann. of Math. (2) **47**, 460–475 (1946); these Rev. **8**, 50]. Further connections are obtained in this paper and numerous specializations are stated. Of particular interest is the following: if Y is a 2-sphere and G^2_m is the component of G^2 of maps of degree m, then $\pi_1(G^2_m)$ is cyclic of order $2m$. Hence no two components of G^2 have the same homotopy type.

N. E. Steenrod (Princeton, N. J.).

Hu, Sze-tsen. Homotopy properties of the space of continuous paths. Portugaliae Math. **5**, 219–231 (1946).

Let X be an arcwise connected separable metric space, $[X, B, a]$ the space of all paths in X whose initial point is a and whose terminal point lies in a subset B of X. The author studies the homotopy properties of $[X, B, a]$, and proves, in particular, that the homotopy type of $[X, B, a]$ does not depend upon a and is unchanged under certain admissible changes (deformations) of B. Let, furthermore, Ω be the arc-component of a path σ in $[X, B, a]$ (B supposed to be arcwise connected) and Ω_0 the arc-component of σ in $[X, b, a]$, where b is the terminal point of σ; natural relations are established between the absolute and relative homotopy groups of Ω and Ω_0, and those of X and B; and n-simplicity [cf. Eilenberg, Fund. Math. **32**, 167–175 (1939)] is investigated. The proofs are given by explicit construction of all deformations involved. *B. Eckmann.*

Radó, T., and Reichelderfer, P. On cyclic transitivity. Fund. Math. **34**, 14–29 (1947).

The paper was originally to have appeared in Fund. Math. **33**. During the suspension of publication of this journal, it was published in Duke Math. J. **6**, 474–485 (1940); these Rev. **1**, 318.

Gottschalk, W. H. On k-to-1 transformations. Bull. Amer. Math. Soc. **53**, 168–169 (1947).

Given a compact Hausdorff space X, for $x\epsilon X$ let $o(x)$ be the Menger order of x, and suppose $f(X)=Y$ is a mapping (i.e., continuous single-valued transformation). If for each $y\epsilon Y$ the set $f^{-1}(y)$ contains k or fewer points (exactly k points) then f is at most k-to-1 (exactly k-to-1). Theorem 1. If f is at most k-to-1 and if the inverse points of $y\epsilon Y$ are $x_1, \cdots, x_n$, then $\sum_{i=1}^n o(x_i) \leqq ko(y)$. Theorem 2. If f is exactly k-to-1 on the continuum X but is not exactly k-to-1 on any proper subcontinuum of X then Y has no end point; if furthermore $k=2$, then Y has no cut point.

J. H. Roberts (Durham, N. C.).

Roberts, J. H. Open transformations and dimension. Bull. Amer. Math. Soc. **53**, 176–178 (1947).

The author considers open transformations, not necessarily continuous, with range and domain metric separable spaces. The results are as follows. (1) Any space A of positive dimension is the domain of a one-to-one open transformation to a space B with $\dim B = \dim A - 1$. (2) If f is open and carries A onto a locally compact space B, and if, for each $y\epsilon B$, $f^{-1}(y)$ is not dense in itself, then $\dim B \leqq \dim A$. It was shown by P. Alexandroff [C. R. (Doklady) Acad. Sci. URSS (N.S.) **13**(1936 IV), 295–299 (1936)] that the conclusion of (2) holds if A is compact and $f^{-1}(y)$ is always countable. However, (3) there exist countable-fold open mappings which increase dimension. This result is obtained by applying to a known dimension-raising map the theorem: (4) if f is open, carrying A onto B, then there is a subspace of A whose image is all of B and on which f is open and countable-fold. Finally, an example is given of a one-to-one open map which carries a 0-dimensional plane set onto a 1-dimensional plane set. *J. L. Kelley* (Chicago, Ill.).

GEOMETRY

★Robinson, G. de B. The foundations of geometry. Proc. First Canadian Math. Congress, Montreal, 1945, pp. 241–251. University of Toronto Press, Toronto, 1946. $3.25.

Expository lecture.

Buchner, P. Eine Aufgabe, die mit Zirkel und Lineal nicht lösbar ist. Elemente der Math. **2**, 14–16 (1947).

The problem of constructing a triangle if two sides and the radius of the inscribed circle are given leads to an irreducible cubic equation for $s=\frac{1}{2}(a+b+c)$; a construction by means of ruler and compass only is therefore impossible.

F. A. Behrend (Melbourne).

★Fejes, László. Extremal distributions of points in the plane, on the surface of the sphere and in space. Univ. Francisco-Josephina. Kolozsvár. Acta Sci. Math. Nat., no. 23, iv+54 pp. (1944). (Hungarian)

The author investigates various problems about densest distribution of points in the plane, surface of the sphere or space. Among others he investigates the following problem. What is the maximum number of points one can place in a given area in the plane so that the distance between any two should be greater than ϵ? The same problem is also investigated on the surface of the sphere and in space. Asymptotic formulas are obtained in the first two cases but the third is still unsolved. Also in case of the plane and sphere asymptotic formulas are found for the minimum number of spheres of radius ϵ covering a given area. In space the problem is again unsolved. Analogous problems are investigated for ellipses instead of circles and various other extremal problems are investigated and many unsolved questions are raised. Some of the questions dealt with in this monograph have previously been investigated by Fejes and others, but the paper contains much new material. *P. Erdös* (Syracuse, N. Y.).

Lesavre, Jean, et Mercier, Raymond. Dix nouveaux polyèdres semi-réguliers, sans plan de symétrie. C. R. Acad. Sci. Paris **224**, 785–786 (1947).

Let $s\{{p \atop q}\}$ denote a polyhedron having each vertex surrounded by the polygons $\{3\}$, $\{p\}$, $\{3\}$, $\{q\}$, $\{3\}$, where $\{p\}$ denotes a regular p-gon when p is an integer (e.g., an equilateral triangle when $p=3$) and a pentagram or star-pentagon when $p=5/2$. This polyhedron exists whenever p and q are two of 3, 3, 4 or two of 3, 5, 5/2. The further ends of the five edges at one vertex form a plane pentagon called the vertex figure, whose sides are 1, b, 1, c, 1, where $b=2\cos\pi/p$, $c=2\cos\pi/q$. This pentagon is inscribed in a circle at whose center each unit side subtends an angle $2\arccos\frac{1}{2}\lambda$, where λ is the positive root of the quartic

equation $x^4+bcx^3-2x^2-3bcx-(b^2+c^2-1)=0$. The ratio of edge to circumradius is easily computed in the form $a/R=2\{1-1/(4-\lambda^2)\}^{\frac{1}{2}}$.

When $p=3$, so that each vertex is surrounded by four triangles and a $\{q\}$, we have $b=1$, and after removal of an extraneous factor $x+c$ the equation for λ reduces to $x^3-2x-c=0$. In particular, $s\{{3 \atop 3}\}$ is the regular icosahedron, while $s\{{3 \atop 4}\}$ and $s\{{3 \atop 5}\}$ are two of the Archimedean solids: Kepler's "snub cube" and "snub dodecahedron." There are also star-polyhedra $s\{{5 \atop 5/2}\}$ and $s'\{{5 \atop 5/2}\}$, related to the regular Kepler-Poinsot polyhedra. The equation $x^3-2x-c=0$ has three real roots if $c<(32/27)^{\frac{1}{2}}$; e.g., when $q=5/2$ the roots are approximately 1.5489, -1.2225, -0.3264. The negative roots can be interpreted by allowing the vertex figure to be nonconvex. The corresponding polyhedra, say $s\{{3 \atop 5/2}\}$, $s'\{{3 \atop 5/2}\}$, $s''\{{3 \atop 5/2}\}$ all have 60 vertices, 150 edges, 60+20 triangles and 12 pentagrams. The values of a/R are 1.2253, 1.5503, 1.7241.

Finally, when $p=5$ and $q=5/2$, the equation for λ is $x^4+x^3-2x^2-3x-2=0$, which has just two real roots: 1.67871 and -1.57289. The corresponding polyhedra, $s\{{5 \atop 5/2}\}$ and $s'\{{5 \atop 5/2}\}$, both have 60 vertices, 150 edges, 60 triangles, 12 pentagons and 12 pentagrams. The values of a/R are 0.7846 and 1.1742.

The authors have rediscovered these five star-polyhedra, which were described by J. C. P. Miller in his dissertation at Cambridge in 1930 [unpublished]. The title of the article comes from the observation that these, and the two classical "snubs," are the only known "uniform" polyhedra whose symmetry group consists entirely of rotations, so that each figure occurs in two enantiomorphous varieties. The authors remark, furthermore, that $s\{{5 \atop 5/2}\}$, $s\{{3 \atop 5/2}\}$ and $s''\{{3 \atop 5/2}\}$ have densities 3, 7 and 37; but the concept of density is less precise in the remaining two cases, because of the "figure of eight" shape of the vertex figures. *H. S. M. Coxeter.*

✱**Coxeter, H. S. M. The nine regular solids.** Proc. First Canadian Math. Congress, Montreal, 1945, pp. 252–264. University of Toronto Press, Toronto, 1946. $3.25.

The five Platonic solids may be denoted by $\{3, 3\}$, $\{3, 4\}$, $\{4, 3\}$, $\{3, 5\}$ and $\{5, 3\}$, where $\{p, q\}$ denotes a regular polyhedron having N_1 edges, p on each face and q at each vertex. The planes of symmetry of $\{p, q\}$ cut out $4N_1$ spherical triangles $(p, q, 2)$ of angles π/p, π/q, $\pi/2$ on the circumsphere. By using three mirrors in the planes of the sides of one triangle, the polyhedral kaleidoscope of Möbius reproduces the whole network. More generally, if an arbitrary spherical triangle (p, q, r) is the smallest fundamental region for a group of order g, generated by reflections, then $4/g=1/p+1/q+1/r-1>0$. The five integral solutions are $(p, p, 1)$, $(p, 2, 2)$, $(3, 3, 2)$, $(4, 3, 2)$, $(5, 3, 2)$. Using a larger fundamental region with angles not exceeding $\pi/2$, the images may cover the sphere D times. Thus four regular polyhedra are found for which either p or q is 5/2, no other fractional values being possible. These are the great dodecahedron and great icosahedron of Kepler and the small and great stellated dodecahedrons of Poinsot. Regular figures may be enumerated for Euclidean n-space or for hyperbolic space by using these same ideas about the fundamental region of the symmetry group. *J. S. Frame.*

Gormley, P. G. Stereographic projection and the linear fractional group of transformations of quaternions. Proc. Roy. Irish Acad. Sect. A. 51, 67–85 (1947).

The author generalizes the classical representation of the complex number $(x+yi)/(1+z)$ by the point (x, y, z) on the 2-sphere $x^2+y^2+z^2=1$, so as to obtain a representation of the quaternion $q=(x_0+x_1e_1+x_2e_2+x_3e_3)/(1+x_4)$ by the point $(x_0, x_1, x_2, x_3, x_4)$ on the 4-sphere $x_0^2+\cdots+x_4^2=1$ in Euclidean 5-space. He finds that the general displacement of this sphere (into itself), being the product of an even number of reflections in hyperplanes through the center, is given by the "unitary" transformation

$$q'=(aq+b)(cq+d)^{-1}=(\bar{a}-q\bar{b})^{-1}(q\bar{d}-\bar{c}),$$

where $\bar{a}a+\bar{c}c=\bar{b}b+\bar{d}d$ and $\bar{a}b+\bar{c}d=\bar{b}a+\bar{d}c=0$. In particular, the pure rotations are $q'=aqa$, $q'=aq\bar{a}$, $q'=-q^{-1}$, and $q'=(q-u)(\bar{u}q+1)^{-1}=(q\bar{u}+1)^{-1}(q-u)$, the last being a rotation through angle 2 arctan $|u|$.

Projecting stereographically from $(0, 0, 0, 0, -1)$, the author obtains the natural representation of

$$q=q_0+q_1e_1+q_2e_2+q_3e_3$$

by the point (q_0, q_1, q_2, q_3) in 4-space. The reflection in a hyperplane through the point of projection remains a reflection when we pass to the 4-space, so it is not surprising that this part of the work [pp. 68–71] duplicates the reviewer's [Amer. Math. Monthly 53, 136–146 (1946); these Rev. 7, 387]. However, the author goes further by considering the group consisting of products of even numbers of inversions in 3-spheres. The general operation of this group is found to be $q'=(aq+b)(cq+d)^{-1}=(a'-qc')^{-1}(qd'-b')$, where

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Finally, replacing the sphere $x_0^2+x_1^2+x_2^2+x_3^2+x_4^2=1$ by the hyperboloid $-x_0^2-x_1^2-x_2^2-x_3^2+x_4^2=1$, he obtains the Minkowskian rotation $q'=(q+u)(\bar{u}q+1)^{-1}$, which reduces to a Lorentz transformation when we set $x_0=0$ and take q and u to be pure quaternions (or vectors). *H. S. M. Coxeter* (Toronto, Ont.).

Lauffer, R. Ein System von acht Symmetriedreiecken eines euklidischen Dreieckes. Akad. Wiss. Wien, S.-B. IIa. 151, 277–291 (1942).

In the triangle ABC, let m_1 be the perpendicular to BC at its midpoint, etc., and let w_1, w_1' be the bisectors of A, etc. The 12 points $[mw]$ and $[mw']$ are considered; the author forms 8 triangles from them, each associated with a certain orientation on each side of ABC. The configuration of the 8 "triangles of symmetry" is studied in connection with the elliptic motion known in kinematics. *O. Bottema* (Delft).

Thébault, Victor. Sur la géométrie récente du tétraèdre. C. R. Acad. Sci. Paris 224, 1267–1269 (1947).

Thébault, V. Sur la géométrie récente du tétraèdre. Ann. Soc. Sci. Bruxelles. Sér. I. 61, 12–17 (1947).

Goormaghtigh, R. Orthopolar and isopolar lines in the cyclic quadrilateral. Amer. Math. Monthly 54, 211–214 (1947).

Pascali, Justo. Projective generation of W curves. Publ. Inst. Mat. Univ. Nac. Litoral 6, 25–39 (1946). (Spanish) [MF 16912]

The greater part of this paper is devoted to a solution of the following problem. Given a triangle XYZ, to determine the plane curves such that, if P' is the point in which the line through Z and the typical point P of the curve meets the line XY, and if P'' is the point in which the tangent to the curve at P meets XY, then the point rows generated by

P' and P'', as P moves along the curve, are related by a projectivity in which X and Y are self-corresponding points. The curves in question are determined explicitly for the cases in which X and Y are real and distinct, real and coincident, and the circular points of the plane. A short discussion is given of the modified problem in which the point P'' is determined, not by the tangent, but by the normal to the curve at P. *L. A. MacColl* (New York, N. Y.).

Glagolev, A. A. On conjugateness of two triplets of points. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 291–292 (1946).

This note begins with a definition. Let ABC, $A'B'C'$ be the triangles formed by the tangents to a given conic C^2 at points X_r, X_r', respectively ($r=1, 2, 3$), and let D, D' be their polar points with respect to a conic M^2. Then the triplets of points X_r, X_r' are called conjugate to one another with respect to M^2 if $ABCDA'B'C'D'$ lie on a conic. (If M^2 coincides with C^2, then D, D' are the Brianchon points of the triangles). The author then states a series of eight theorems. For example, all triplets of C^2 conjugate, with respect to M^2, to a given triplet X_r, form a cubic involution I_3^2; all triplets conjugate to given triplets X_r, Y_r form an involution I_3^1, such that any triplet of I_3^2 is conjugate (with respect to M^2) to any triplet of I_3^1; the collineation that can be established between the straight lines of space and the involutions I_3^1 on C^2 associates with the self-conjugate involutions I_3^1 the rays of a linear complex Γ; and so on. The theorems develop the ideas of Appell [Sur les propriétés des cubiques gauches, thesis, Paris, 1876] and of Vlasov [Učenyya Zapiski Imp. Mosk. Univ. Otd. Fiz.-Mat. **25** (1911); see Jahrbuch über die Fortschritte der Math. **41**, 608]. *H. S. Ruse* (Leeds).

Hohenberg, Fritz. Eineindeutige involutorische Kegelschnittverwandtschaften, die sich mit Hilfe eines festen Kegelschnitts definieren lassen. Akad. Wiss. Wien, S.-B. IIa. **152**, 15–101 (1944).

The author studies involutory one-to-one correspondences between conics which can be defined by means of a fixed conic k. The investigation is restricted to the system (S) of conics which have the same self-polar triangle in common with k. The general correspondences are obtained by applying the automorphic collineations of k to the involutions in this system. The system (S) of conics is mapped on an auxiliary plane. A fixed line is given; each conic α of the system is mapped on the pole of this line with respect to α. Thus every involutory one-to-one correspondence between conics of (S) is transformed into an involutory one-to-one correspondence between points. This mapping is the principal tool used in this study.

In the first part of the paper three special correspondences are discussed. (1) The polarity with respect to the fixed conic k. (2) The involution in which two conics correspond if the fixed conic k is their harmonic curve of second order (i.e., the common tangents of corresponding conics touch them in 8 points which lie also on k). (3) The involution in which two conics correspond if the fixed conic k is their harmonic curve of class two (i.e., the 8 tangents to corresponding conics, drawn at their points of intersection, are also tangents of k). In the second part of the paper more general involutions between conics are considered. It is shown that any involutory one-to-one correspondence between conics is mapped on a Geiser-involution. The proof is made by using Bertini's reduction of point involutions to four independent types [Ann. Mat. Pura Appl. (2) **8**, 244–286 (1877); Ist. Lombardo Sci. Lett. Rend. (2) **13**, 443–452 (1880)]. A detailed discussion of the corresponding involutions between conics is given. The case of systems of conics tangent to k or osculating k is also considered. *E. Lukacs* (Cincinnati, Ohio).

Mateo, José. Construction of conics deduced from a particular property with respect to a circle, and a criterion for determining their type. Math. Notae **6**, 96–111 (1946). (Spanish)

Two conics γ_1 and γ_2 are said to be "in the relation ω" if the two projectively related pencils which generate γ_1 and have their vertices at a pair of common points of the conics cut γ_2 in pairs of points of an involution. Some properties of this relationship are discussed and it is shown that it is symmetrical in the two conics. Though any two points P and Q of γ_1 we can draw a circle γ_2 in the relation ω to γ_1, and the relative position of this circle and the pole of PQ with respect to γ_1 determines whether γ_1 is a circle, ellipse, parabola or hyperbola. The circle is used for various geometrical constructions [e.g., given three points of γ_1 and the tangents at two of them we can draw γ_2 and then with the aid of γ_2 and a ruler we can construct all the points on, and the tangents to, γ_2 as well as the pairs of conjugate diameters]. *D. B. Scott* (Aberdeen).

Levi, Beppo. Properties of the base quadrangle of a pencil of conics. Math. Notae **6**, 112–115 (1946). (Spanish)

It is pointed out that the "relation ω" of the paper reviewed above (which Levi calls pseudo-orthogonality) is a projective generalisation of the relationship of a pair of orthogonal circles. In any pencil of conics with distinct base points there are three involutions of pseudo-orthogonal pairs, associated, respectively, with the sides of the common self-conjugate triangle. A reference is given to Baker [Principles of Geometry, vol. 2, Cambridge University Press, 1930, pp. 42–44], who shows that the property is symmetrical, and also to Salmon [Sections Coniques, 3d ed., p. 595] who considers the equivalent relationship between two conics whose harmonic envelope is degenerate. *D. B. Scott* (Aberdeen).

Turri, Tullio. La continuità delle trasformazioni conservanti i gruppi armonici. Rend. Sem. Fac. Sci. Univ. Cagliari **15**, 90–97 (1946).

On considère dans cet article les transformations T de la droite complexe conservant les groupes harmoniques et laissant fixes les points 0, 1, ∞. Il y est montré que la transformation T laisse la transformation des conjugés ($z'=\bar{z}$) invariante. L'auteur, dans sa démonstration, utilise de façon essentielle la propriété qu'a T de faire correspondre aux sommets opposés d'un carré du plan de Cauchy les sommets opposés d'un autre carré. S'appuyant sur certains résultats établis par É. Cartan [Leçons sur la Géométrie Projective Complexe, Gauthier-Villars, Paris, 1931], il montre que toute transformation de la droite complexe conservant les groupes harmoniques est une homographie ou une antihomographie, et est par suite continue. *P. Vincensini.*

Nyström, E. J. Die Projektivität der Ebene. Soc. Sci. Fenn. Comment. Phys.-Math. **13**, no. 4, 8 pp. (1 plate) (1945).

The author classifies the different types of plane collineations according to the number of parameters involved. *E. Lukacs* (Cincinnati, Ohio).

Bruins, E. M. The projective invariants of four G_d's in G_{2d}. Nederl. Akad. Wetensch., Proc. **49**, 738–743 = Indagationes Math. **8**, 441–446 (1946). (Dutch)

The author shows that, for four $[d-1]$'s in $[2d-1]$, symbolically given by a^d, α^d, p^d, π^d, a complete system of invariants exists, containing the following invariants: six of the type $A_{12}=(a^d\alpha^d)$ and the $d-2$ invariants

$$I_{1234}^{(d-\lambda)}=(a^d\alpha^{d-\lambda}p^\lambda)(p^{d-\lambda}\alpha^\lambda\pi^d), \qquad \lambda=1, 2, \cdots, d-2.$$

O. Bottema (Delft).

Kollros, Louis. Théorème de l'hyperespace analogue au théorème de Pascal. Comment. Math. Helv. **19**, 316–319 (1947).

A quadratic hypersurface and a simplex are given in an n-dimensional space. The edges of the simplex meet the hypersurface in $n(n+1)$ points. These can be arranged in $n+1$ groups of n points so that each group contains points on n concurrent edges. The $n+1$ groups determine $n+1$ hyperplanes E_{n-1}. One of these hyperplanes corresponds to each vertex of the simplex and to its opposite face. Each of these hyperplanes intersects the corresponding face of the given simplex in an E_{n-2}. It is shown that these $n+1$ linear $(n-2)$-dimensional spaces are associated. Here $n+1$ linear $(n-2)$-dimensional spaces are said to be associated if any straight line which intersects n of them also meets the last. The dual theorem is also stated. For $n=2$ one obtains Pascal's and Brianchon's classical theorems. For $n=3$ the theorem, as well as the dual proposition, were known to Chasles. *E. Lukacs* (Cincinnati, Ohio).

***Hjelmslev, Johannes. Contact invariants.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 241–244. Jul. Gjellerups Forlag, Copenhagen, 1947. (Danish)

An account of the author's work [Danske Vid. Selsk. Math.-Fys. Medd. **20**, no. 21 (1943); these Rev. **7**, 388].

Savage, L. J. The application of vectorial methods to metric geometry. Duke Math. J. **13**, 521–528 (1946).

L'article est d'abord consacré à quelques propositions sur les espaces vectoriels V munis d'un produit scalaire (v_1, v_2) non nécessairement positivement défini, généralisant sous forme géométrique les résultats classiques sur la réduction d'une forme quadratique. Soit $A(V)$ l'ensemble des éléments de V qui sont orthogonaux à tout élément de V; V est dit non dégénéré si $A(V)=\{0\}$; dans le cas général, le quotient V^* de V par $A(V)$, appelé espace réduit, est non dégénéré. L'auteur appelle "distance space" M un ensemble d'éléments $p, q, r, s, \cdots$ tel que soit définie sur les couples (p, q) une fonction à valeurs réelles $\sigma(p, q)$ satisfaisant aux conditions $\sigma(p, p)=0$, $\sigma(p, q)=\sigma(q, p)$. Un espace métrique, où $\rho(p, q)$ représente la distance de p à q, est à considérer comme "distance space" en posant $\sigma(p, q)=\rho^2(p, q)$.

Supposons que g soit une isométrie d'un espace M dans un espace V, c'est-à-dire que $\sigma(p, q)=(g(p)-g(q), g(p)-g(q))$; à tout couple ordonné (p, q) de points de M, nous faisons correspondre $\overrightarrow{pq}=g(p)-g(q)$, alors pour deux couples (p, q) et (r, s)

(E) $\quad (\overrightarrow{pq}, \overrightarrow{rs})=\frac{1}{2}\{\sigma(p, s)+\sigma(q, r)-\sigma(p, s)-\sigma(q, s)\}.$

Le résultat central de l'article s'énonce ainsi. Désignons par $W(M)$ l'espace linéaire sous-tendu par $g(M)$, par $V(M)$ l'espace réduit correspondant; $V(M)$ est défini à une isomorphie près par la donnée de M; un modèle abstrait en est le suivant. A tout couple (p, q) de M nous attribuons un élément $\overrightarrow{pq}$ et nous considérons comme vecteurs les combinaisons linéaires à coefficients réels des vecteurs $\overrightarrow{pq}$, $x=\sum\alpha_{pq}\cdot\overrightarrow{pq}$; le produit scalaire de $\overrightarrow{pq}$ par $\overrightarrow{rs}$ est défini par (E) et le produit scalaire de deux vecteurs x et y par prolongement linéaire. L'espace réduit de l'espace ainsi obtenu est $V(M)$ à une isomorphie près. La correspondance entre (pq) et $\overrightarrow{pq}$ satisfait à $\overrightarrow{pq}+\overrightarrow{qr}=\overrightarrow{pr}$. Les propriétés d'immersion de M dans un espace V peuvent alors s'exprimer au moyen des propriétés de $V(M)$. Ainsi une condition nécessaire et suffisante pour que M puisse être plongé dans un espace hilbertien est que $V(M)$ soit défini positif, d'où découle aisément le critère de Menger: un espace M est plongeable dans un espace hilbertien dans le cas et seulement dans le cas où tout système fini de ses points est euclidien. Le critère de W. A. Wilson [Amer. J. Math. **57**, 322–326 (1935)] est généralisé. *C. Pauc* (Marseille).

Arvesen, Ole Peder. Zur analytischen Lösung der Pohlkeschen Aufgabe. Norske Vid. Selsk. Forh., Trondhjem **14**, no. 23, 87–89 (1941).

Cf. Norsk Mat. Tidskr. **23**, 100–108 (1941); these Rev. **8**, 220.

Giovanardi, M. Sulla prospettiva (conica e cilindrica) delle superficie di rotazione. Rend. Accad. Sci. Fis. Mat. Napoli (4) **9**, 99–107 (1939).

Convex Domains, Integral Geometry

Alexandroff, A. D. On the gluing of convex surfaces. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 99–101 (1946).

Let $P_1, P_2, \cdots$ be closed pieces of convex surfaces in the three-dimensional Euclidean (or hyperbolic or spherical) space. The P_i are considered as polygons. It is assumed that the edges are rectifiable and have semitangents at their endpoints. The concept of total (or integral of the) geodesic curvature can be generalized to these edges [see the expression $\tau_D(C)$ in the same C. R. (N.S.) **47**, 315–317 (1945); these Rev. **7**, 167]. The total geodesic curvature is assumed to be a function of bounded variation on the subsegments of an edge.

Let the edges and vertices of the P_i be identified so as to form a manifold M (possibly with a boundary) and so that the mapping of corresponding edges on each other is length preserving. In order that M be locally isometric to convex surfaces it is necessary and sufficient that the sum of the angles [defined as in the paper cited above] at one vertex does not exceed 2π and that the sum of the two total geodesic curvatures corresponding to the same edge is nonnegative. The manifold M is isometric to a closed convex surface if it is homeomorphic to a sphere. No proofs are given.

H. Busemann (Los Angeles, Calif.).

Hadwiger, H. Über eine symbolisch-topologische Formel. Elemente der Math. **2**, 35–41 (1947).

Let $K_1, \cdots, K_n$ be bounded closed convex sets in the Euclidean plane E. Put $A=\cup_i K_i$ and denote by $z(A)$ and $z(E-A)$ the number of (maximal connected) components of A and $E-A$, respectively. Then

(*) $\quad (E-K_1)(E-K_2)\cdots(E-K_n)=z(E-A)-z(A),$

where the product on the left is to be developed algebraically and every summand is to be replaced by 0 or 1 according as the corresponding set is empty or not. The relation (*)

contains many well-known facts: for instance, Euler's formula for polyhedra, the Radon-Helly theorem in the plane, theorem on graphs by König and Listing and also the following analogue to a theorem of Steinitz on the decomposition of a plane by straight lines. If the intersection of any three K_i is empty, but any two K_i intersect, then A decomposes E into $(n^2-3n+4)/2$ domains.
H. Busemann (Los Angeles, Calif.).

Hadwiger, H. Ueber die erweiterten Steinerschen Formeln für Parallelmengen. Revista Mat. Hisp.-Amer. (4) 6, 160–163 (1946).

Après avoir rappelé une définition donnée antérieurement par lui [Comment. Math. Helv. 18, 59–72 (1945); ces Rev. 7, 260] d'une sous-convexité et d'une surconvexité particulières [cette dernière ne coïncidant pas avec celle de Mayer], l'auteur utilise la formule cinématique fondamentale de Blaschke [Vorlesungen über Integralgeometrie, Leipzig-Berlin, 1936, p. 37] pour établir le théorème suivant. Si le domaine plan fermé et borné A est sous-convexe d'indice α et surconvexe d'indice β, on a dans l'intervalle $-\beta<\rho<\alpha$

$$F_\rho=F+\rho L+(n-m)\pi\rho^2,\quad L_\rho=L+2(n-m)\pi\rho,$$

où F, L, F_ρ, L_ρ sont la surface du domaine, la longueur de sa frontière, la surface et la longueur correspondante pour le domaine parallèle d'indice ρ, n est le nombre des frontières extérieures de A, m celui de ses frontières intérieures. Ces formules généralisent des formules classiques de Steiner.
E. Blanc (Clermont-Ferrand).

Vidal Abascal, E. On a theorem of Liouville and generalization of Steiner's formulas. Revista Mat. Hisp.-Amer. (4) 6, 254–259 (1946). (Spanish)

Two new proofs of Liouville's theorem that a surface with two families of geodesics which intersect at constant angles is developable. Also, Steiner's formulae for length L_ρ and area F_ρ of a parallel curve at distance ρ to a closed curve (in the plane) of length L and area F are proved again. The author states the following generalization of these formulae to surfaces of constant negative curvature $-\beta^2$:

$$L_\rho=L\cosh\beta\rho+2\pi\beta^{-1}\sinh\beta\rho-\beta F\sinh\beta\rho,$$
$$F_\rho=L\beta^{-1}\sinh\beta\rho+2\pi\beta^{-2}(\cosh\beta\rho-1)+F\cosh\beta\rho$$

and the corresponding formula for positive curvature β^2 [which is due to Hadwiger, Comment. Math. Helv. 18, 59–72 (1945); these Rev. 7, 260]. *H. Busemann.*

Inzinger, Rudolf. Über Mittelpunktseilinien. Akad. Wiss. Wien, S.-B. IIa. 155, 1–14 (1946).

The oriented lines in a plane P_1 can be mapped on the pairs of nonoriented lines symmetric with respect to a fixed point O_2 in a plane P_2 in such a way that every oriented circle in P_1 goes into a conic section with center O_2 in P_2. An oval with O_2 as center is the image of a curve K_1 in P_1 which is inversely convex with respect to the original O_1 of O_2; this means that K_1 contains O_1 in its interior and K_1, as well as its inverse with respect to O_1, is convex.

The mapping is used to deduce various results on ovals with centers from facts on inversely convex ovals and conversely. The following results are typical. Let $Z(K_1)$ be the set of all these points with respect to which K_1 is inversely convex; $Z(K_1)$ consists of these points which lie in the interior of all osculating circles of K_1 and lies in the "Inkreis" of K_1 and also in the interior of the inner circle of the "Minimalkreisring" of K_1. [These terms are defined by Bonnesen and Fenchel, Theorie der konvexen Körper, Ergebnisse der Math., v. 3, no. 1, Springer, Berlin, 1934]. The conic order K of an oval K_2 with center O_2 is the maximal number of pairs of diametrically opposite points in which a conic section with center O_2 can intersect K_2. The number of ellipses which hyperosculate K_2 in diametrically opposite points is at least K. *H. Busemann.*

✶Scherk, Peter. The four-vertex theorem. Proc. First Canadian Math. Congress, Montreal, 1945, pp. 97–102. University of Toronto Press, Toronto, 1946. $3.25. on order

The classical four-vertex theorem concerns the number of extrema of the curvature on a convex plane curve. In this report on recent and potential developments the author seeks to show that the theorem may be freed of much of the paraphernalia of Euclidean differential geometry and appear in a more general form. Results by Jackson [Bull. Amer. Math. Soc. 50, 564–578 (1944); these Rev. 6, 100] show that the theorem belongs at least to conformal geometry, since vertices may be defined as points where the osculating circle is a circle of support and this property is invariant under circular transformations. By stereographic projection, these results may be interpreted as theorems on spherical curves. Mohrmann [S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1917, 1–4] has gone still further in this direction by considering curves on a general ovaloid and establishing results which include a considerable amount of later work as a direct corollary. If the circles of the four-vertex theorem are replaced by other families of curves satisfying suitable restrictions, vertices may be defined relative to such families, thus establishing further generalizations of the theorem. This has been done by Haupt [Monatsh. Math. Phys. 40, 1–53 (1933)]. From this point of view the theorem appears to be of an essentially topological nature.
S. B. Jackson (College Park, Md.).

Dinghas, Alexander. Beweis der isoperimetrischen Eigenschaft der Kugel für den n-dimensionalen Raum. Akad. Wiss. Wien, S.-B. IIa. 149, 399–432 (1940).

The author gives a proof of the isoperimetric inequality in n dimensions by a method applicable directly to a class of bodies which are not necessarily convex, without using any symmetrizations or convex-rendering operations. The proof is based on a sharpened form of the Brunn-Minkowski inequality: if F is a closed set in the cube $|x_i|\leqq a$, $i=1,\cdots,n$, whose Jordan content F exists, if F_0 is the unit n-sphere and if F_h is the linear combination $F+hF_0$, then $F_h\geqq(F^{1/n}+hF_0^{1/n})^n+hc_nJ_n^2$. Here $J_n^2=\int\{S(P)CS(P)\}^{\frac{1}{2}}dP$, where the integral is extended over the $(n-1)$-plane $x_1=0$, $S(P)$ is the Lebesgue linear measure of the set F_P in which F intersects a line orthogonal to $x_1=0$, and $CS(P)$ is the measure of the complement of F_P with respect to the minimum segment containing F_P; c_n is a number depending only on a and n. If in addition the Minkowski surface T of F exists, and if the set of interior points of F is nonnull, connected and dense in the boundary of F, the above inequality leads directly, on allowing h to approach zero, to $T\geqq nF^{1-1/n}F_0^{1/n}+c_nJ_n^2$, which is a sharpened form of the isoperimetric inequality. An examination of the sharpening term $c_nJ_n^2$ yields a proof that the isoperimetric equality holds only for a sphere. *J. W. Green.*

Vahlen, Theodor. Isoperimetrie, differenzengeometrisch. Deutsche Math. 7, 368–373 (1944).

Let K be a convex polygon of area S and perimeter T. If all sides of K are displaced outwards a distance x, a

convex polygon K' is obtained, with area S' and perimeter T'. Let E be the total area of the quadrilaterals at the vertices of K in the case $x=1$; each quadrilateral is bounded by two sides of K' and the two normals to the sides of K at the vertex. It is readily shown that $E>\pi$ and that (1) $S'=S+Tx+Ex^2$, $T'=T+2Ex$, $E=E'$. From equations (1) it follows that $(T')^2-4E'S'=T^2-4ES$. If the sides are displaced inwards, equations (1) still hold, with $-x$ replacing x, as long as the displacement does not cause the loss of any side of K. The author analyses the effect of loss of sides and sketches a proof that $(T')^2-4E'S'$ does not increase. Eventually by such inward displacement one arrives at either a rectangle or a polygon circumscribable about a circle; for these it is possible to prove directly that $(T')^2-4E'S'\geqq 0$. Hence $T^2-4ES\geqq 0$; this, together with $E>\pi$, gives the isoperimetric inequality for convex polygons. A corresponding analysis gives the isoperimetric inequality for convex polyhedra. *J. W. Green* (Los Angeles, Calif.).

Ventikos, Gr. P. On the mean value of a straight segment in a plane convex domain. Bull. Soc. Math. Grèce 22, 195–197 (1946). (Greek)

Let C be a domain of area F in E^2 bounded by a closed convex curve of length L. The number N (in the sense of integral geometry) of directed segments $\bar{\alpha}$ of length α contained in C satisfies the inequality $2(\pi F-\alpha L)<N<2\pi F$. If

$$d\alpha\int_{\bar{\alpha}\subset C} dxdyd\theta,$$

where θ is the angle of $\bar{\alpha}$ with the x-axis, is taken as density differential for the set of all segments of length α in C and D is the diameter of C, then the mean value $[\alpha]$ of the lengths of all segments in C satisfies the relation

$$F/2\pi D<[\alpha]<F^2/(2\pi FD-LD^2).$$

The right hand inequality holds whenever $2\pi F>DL$.
H. Busemann (Los Angeles, Calif.).

Differential Geometry

De Cicco, John. Cartography and scale curves. Revista Unión Mat. Argentina 12, 62–74 (1946). (Spanish)

Exposition of recent results of Kasner and the author.

Samuel, Pierre. Correspondance conforme de deux surfaces à plans tangents parallèles. Ann. Univ. Lyon. Sect. A. (3) 5, 19–29 (1942).

[The author's name was omitted in the journal.] Let S and S' be parallel surfaces; that is, S and S' are surfaces such that the normals to S and S' at corresponding points are parallel. The author studies directly the classical problem of determining conditions under which the parallel correspondence is also conformal. Cases of direct conformality and inverse conformality are treated separately.

In the first case, overlooking the trivial homothetic solutions, the author indicates that the only solutions are minimal surfaces, and shows that any two minimal surfaces can be put in the desired representation. In the second case, it is shown that the class of solutions is the class of isothermic surfaces; that is, the class of surfaces for which the lines of curvature form an isothermic system. When the first surface is given, the second is determined to within a homothetic transformation. *E. F. Beckenbach.*

***Sbrana, Francesco. Sopra alcune proprietà delle superficie.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 313–321. Edizioni Cremonense, Rome, 1942.

Consider an elliptic point O of a surface Σ in ordinary space. Let π be the tangent plane at O, α a plane parallel to π, σ the section of Σ by α, G the centroid of σ and G^* the centroid of the volume enclosed by Σ and σ. It is well known that OG approaches the affine normal at O as α approaches π. While determining this limit in an elementary analytic fashion, the author is led to two theorems: (1) the ellipse of inertia of σ relative to G approaches the Dupin indicatrix at O as α approaches π; (2) OG^* approaches the affine normal at O as α approaches π. *A. Schwartz.*

Rollero, A. Sugli sviluppi canonici di una superficie nell' intorno di un suo punto. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 1055–1059 (1946).

The object of this paper is to obtain a canonical development of a surface S in the neighborhood of one of its points in the form $z=xy+\frac{1}{3}(x^3+y^3)-\frac{1}{18}(x^3+y^3)(Ax+By)+\cdots$, A and B being invariants. The defining differential equations of S being $x_{uu}=\theta_u x_u+\beta x_v+px$, $x_{vv}=\gamma x_u+\theta_v x_v+qx$, the invariants A and B are defined by the expressions $A=\beta^{-1}\gamma^{-1}\varphi$, $B=\beta^{-1}\gamma^{-1}\psi$, $\varphi=(\log\beta\gamma^2)_u$, $\psi=(\log\beta^2\gamma)_v$. One of the vertices of the tetrahedron of reference giving rise to the expansion is the generic point of S and two others are the intersections of the second axis of Bompiani (that is, the canonical line $k=-5/12$) with the asymptotic tangents.
V. G. Grove (East Lansing, Mich.).

Rollero, A. Sul contatto di due rigate lungo una commune generatrice. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 2, 38–41 (1947).

The author proves the following theorems in which R, R' are two nondevelopable ruled surfaces and g a common nonsingular generator. (1) If R, R' have contact of order $s>1$ in three distinct points of g, they have contact of order s in all points of g. (2) If R, R' have contacts of orders s and $s-1$, respectively ($s\geqq 2$), in two distinct points of g they have contact of order $s-1$ in any point of g. (3) If R, R' have contact of order $s\geqq 3$ in a point of g they have contact of order $s-2$ in any point of g. (4) If R, R' have contact of order $s-1$ ($s\geqq 2$) in all points of g there exist two points of g in which they have contact of order s. Theorems 1 and 4 are natural generalizations of well-known consequences of Chasles' theorem for ruled surfaces, while theorems 2 and 3 can be considered as limiting cases of theorem 1.
J. L. Vanderslice (College Park, Md.).

Gheorghiu, Gh. Th. Sur certaines transformations asymptotiques. Disquisit. Math. Phys. 4, 131–173 (1945).

Let S be a surface generated by a point x and S' a surface generated by x'. Let the parametric curves on S be asymptotic, the coordinates x of the generic point satisfying the differential equations $x_{uu}=\theta_u x_u+\beta x_v+px$, $x_{vv}=\gamma x_u+\theta_v x_v+qx$. If x' corresponds to x in such a manner that the parametric curves on S' are asymptotic, then S' is said to be an asymptotic transform of S. A partial solution of the problem of the determination of asymptotic transforms of S is furnished by the congruences of Weingarten, for if one is given a surface S described by x, and if one demands that x' lie in the tangent plane to S at x, then S' will be an asymptotic transform of S if the congruence of lines (x, x') is a W congruence, S and S' being the focal points of the congruence. This paper generalizes the problem by demanding that x' lie on a quadric Q of

Darboux of S at x and that a quadric Q' of Darboux of S' at x' pass through x', S' being an asymptotic transform of S. It is assumed that Q, Q' are determined by the same parameter values, in particular, those which determine the quadric of Lie, or of Fubini. Explicit formulas are given for the coefficients of the defining differential equations of S' in terms of the coefficients of those of S and of the parameter of the quadric of Darboux. Particular attention is paid to the case in which S and S' are projectively applicable and to other particular cases of the asymptotic transformation.

V. G. Grove (East Lansing, Mich.).

Grove, V. G. On congruences and conjugate nets. Amer. J. Math. **69**, 59–69 (1947).

L'article consiste en une étude métrique des réseaux conjugués d'une surface. L'auteur établit un système de formules générales, particulièrement adapté au sujet traité grâce à une normalisation opportune des vecteurs tangents aux courbes du réseau envisagé. Il attache à toute courbe du réseau en chacun de ses points une quantité invariante qu'il appelle la courbure asymptotique de la courbe. Il montre l'analogie entre la courbure asymptotique des courbes d'un réseau conjugué et la courbure géodésique des courbes d'un réseau orthogonal, et établit que si la courbure asymptotique des courbes d'un réseau est constante le long de chaque courbe celui-ci est un reseau conjugué isotherme. Considerant l'intégrale $I=\int(ed_{\alpha\beta}du_\alpha du_\beta)^{\frac{1}{2}}$, où $e=\pm 1$ et $d_{\alpha\beta}$ désigne le deuxième tenseur fondamental, il démontre qu'une courbe est une extrémale pour I si la courbure asymptotique est nulle en chacun de ses points, puis attache à tout réseau isotherme conjugué un faisceau de réseaux dont les courbes sont des extrémales de I, et montre que, conformément à un résultat de Wilczynski, tous les réseaux du faisceau sont conjugués isothermes. Il recherche ensuite la condition pour que la courbure asymptotique soit nulle sur une géodésique, et arrive en particulier à la conclusion que la seule surface de révolution dont les méridiens sont à courbure asymptotique nulle est le caténoïde. La suite traite des conditions pour qu'une congruence, dont les rayons sont issues des points d'une surface S, soit conjuguée à S ou bien soit normale.

P. Vincensini (Besançon).

Vincensini, Paul. Sur une propriété relative à la déformation des surfaces. C. R. Acad. Sci. Paris **224**, 520–522 (1947).

A fuller discussion is made of a net which remains invariant under an arbitrary deformation of its sustaining surface S [same C. R. **222**, 630–632 (1946); these Rev. **7**, 481]. A point I in the tangent plane to S at P corresponds to P and remains invariably in the tangent plane to S at the corresponding point P. The vanishing of the coefficients of the differential equation defining the net imposes conditions on I. For example, if a certain one of the coefficients vanishes then I is the center of geodesic curvature at P of one of the curves of the invariant net.

V. G. Grove.

Finicoff, S. Réseaux conjugués aux axes communs. Bull. [Izvestiya] Math. Mech. Inst. Univ. Tomsk **3**, 75–103 (1946). (Russian. French summary)

Étant donné un réseau conjugué (u, v) tracé sur une surface (M), on appelle le premier axe du réseau la ligne d'intersection des plans osculateurs aux lignes u, v au point M; le second, la droite qui joint les transformés de Laplace M_1, M_2 du point M relativement au réseau (u, v). Cela posé, examinons deux surfaces (M), (M') dont les axes relativement au réseau conjugué commun (u, v) coincident. Le problème se présente en deux espèces selon la coincidence A directe (le premier axe avec le premier axe) ou B inverse (le premier axe de (M) avec le second de (M') et vice-versa).

Deux solutions sont évidentes. (1) Quelle que soit une suite de Laplace périodique à période 4 décrite par le quadrilatère gauche $MM_1M'M_2$, deux réseaux focaux opposés (M) et (M') ou contigus (M) et (M_1) présentent une solution du problème A ou B. (2) Appellons le réseau dont deux congruences appartiennent à des complexes linéaires, réseau de Wilczynski. Cela posé, deux réseaux de Wilczynski (M) et (M') appartenant à la même suite de Laplace présentent une autre solution du problème. Les transformés de Laplace successifs du point M sont situés sur les deux axes du réseau: les transformés d'ordre paire sur le premier axe, ceux d'ordre impaire sur le second. Ils composent avec le réseau (M) des solutions du problème A ou B. Deux congruences des axes du réseau (M) appartiennent à la même congruence linéaire. Une infinité des suites de Wilczynski est attachée à ce couple de congruences, chaque réseau focal compose avec le réseau (M) une solution de A ou de B.

Les configurations citées sont toutes les solutions du problème A. Le problème B admet de plus deux solutions nouvelles. (1) Appellons réseau de Rozet (de v vers u) le réseau dont les points homologues de trois transformés de Laplace successifs dans la direction de u et celui du premier transformé dans la direction de v sont situés dans le même plan. La congruence dont les réseaux focaux sont des réseaux de Rozet dirigés l'un vers l'autre est une congruence la plus générale appartenant à un complexe linéaire. Les réseaux focaux cités présentent une solution de B. (2) Appellons R_1 le réseau R qui se transforme en réseau R (donc R_1) par une congruence d'un complexe linéaire. Les deux réseaux R_1 associés présentent une solution de B. Les congruences des axes appartiennent au même complexe et composent un couple de congruences stratifiables. Les surfaces R qui portent les réseaux R_1 composent une classe spéciale qui dépend de 4 fonctions arbitraires d'un argument. A chaque surface de la classe un seul réseau R_1 est associé et un seul réseau R_1 lui est attaché comme transformé par une congruence d'un complexe linéaire.

Author's summary.

Pa, Chenkuo. A new definition of the Godeaux sequence of quadrics. Amer. J. Math. **69**, 117–120 (1947).

L. Godeaux a montré qu'à tout point d'une surface analytique non développable de l'espace ordinaire est attachée une suite de quadriques φ, φ_1, $\cdots$ dont la première est la quadrique de Lie, telle que deux quadriques consécutives se touchent en quatre points, caractéristiques pour ces deux quadriques [La théorie des surfaces et l'espace réglé, Actual. Sci. Indus., no. 138, Hermann, Paris, 1934]. L'auteur donne à cette suite une signification géométrique. Il suppose la surface rapportée à ses asymptotiques (u, v) et associe à tout point $M(u, v)$ un complexe linéaire $R_1(u, v)$, osculateur en ce point à l'asymptotique u, et à partir de là il obtient la suite de Godeaux. De même, on peut déduire cette suite des complexes linéaires osculateurs aux surfaces réglées R_u engendrées par les tangentes aux courbes v aux différents points des lignes u. Enfin, l'auteur utilise cette nouvelle définition pour obtenir une suite de quadriques associée à tout rayon d'une congruence W.

M. Decuyper (Lille).

Strubecker, Karl. Über die Flächen, deren Asymptotenlinien beider Scharen linearen Komplexen angehören. Anz. Akad. Wiss. Wien. Math.-Nat. Kl. **78**, 90–94 (1941).

After a historical summary the author gives the following theorem. All surfaces Φ whose systems of asymptotic curves

belong to linear complexes are, to within complex projective transformations, the following. (I) The surfaces Φ_I with absolute curvature zero in elliptic space, discovered by Bianchi [Ann. Mat. Pura Appl. (2) **24**, 93–129 (1896). (II) The surfaces Φ_{II} analogous to Φ_I in quasielliptic space, introduced by Blaschke [Ebene Kinematik, Hamburger Math. Einzelschr., no. 25, Teubner, Leipzig-Berlin, 1938, § 15]. (III) The surfaces Φ_{III} with constant relative curvature in isotropic space, studied by Strubecker [Math. Z. **47**, 743–777 (1942); these Rev. **7**, 530]. For the construction of these surfaces there exists a common kinematical method. If in the three spaces considered C_r and C_l are given curves with constant torsion $+1$ and -1, respectively, and if C_r (C_l) is given a right- (left-) hand parallel displacement (in the sense of Clifford) along C_l (C_r), a surface Φ_I, Φ_{II} or Φ_{III} arises to within complex projectivities. The asymptotic lines are the displaced curves. *J. A. Schouten* (Epe).

Urban, Alois. Frenet'sche Formeln der windschiefen Regelflächen. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **43**, 203–205 (1942).

This is a summary of a paper on ruled surfaces in which the osculating linear congruence along a line of the ruled surface is assumed to be merely nonparabolic. Frenet formulas are developed and geometric interpretations of the tensors and scalars appearing in them are given. In "Differentielle Liniengeometrie" by V. Hlavatý [Noordhoff, Groningen, 1945; these Rev. **8**, 346] it was assumed not only that the osculating linear congruence along a line is not parabolic but that the two axes have contact of the same order with the surface along the line [see convention (2.2), p. 73]. *A. Schwartz* (State College, Pa.).

Havlíček, K. Abwickelbare Fläche in der differentialen Liniengeometrie. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **44**, 581–583 (1943).

This is a summary of a paper in which developable ruled surfaces are studied following the methods and symbolism of Hlavatý's "Differentielle Liniengeometrie" [Noordhoff, Groningen, 1945; these Rev. **8**, 346]. The principal theorem states that a necessary and sufficient condition that a ruled surface shall be developable is that its K-map shall be a K-minimal curve, i.e., a curve for which

$$A_{**}\frac{d\xi^*}{dt}\frac{d\xi^*}{dt}=0,$$

where ξ^Σ ($\Sigma=1, 2, 3, 4$) are the independent parameters of the line space and the tensor A_{**} is defined by Hlavatý [p. 472]. The K-minimal curves are shown to be of two types and the corresponding developable surfaces are studied. *A. Schwartz* (State College, Pa.).

Beerten, G., and Van Bouchout, V. Rectilinear congruences with a developable focal surface. Simon Stevin **25**, 33–44 (1947). (Dutch)

Let $\mathbf{f}_a(\xi^1, \xi^2)$, $a=1, 2$, be the ath focal surface of a rectilinear congruence (S). If $\mathbf{s}$ is a unit vector on the generator line S of (S), then (1) $\mathbf{f}_2=\mathbf{f}_1+t\mathbf{s}$, where t is the focal distance on S. If $\mathbf{N}_a$ is the unit normal vector of $\mathbf{f}_a$, then (2) $\mathbf{N}_2=\cos\sigma\, \mathbf{N}_1+\sin\sigma\mathbf{p}$, where $\mathbf{p}$, orthogonal to $\mathbf{s}$, is the tangential plane of $\mathbf{f}_1$. Using the fundamental equations of Gauss and Weingarten for $\mathbf{f}_1$, one may deduce from (1) and (2) the derivatives $\partial\mathbf{f}_2/\partial\xi^\lambda$ and $\partial\mathbf{N}_2/\partial\xi^\lambda$ which give both fundamental tensors of $\mathbf{f}_2$. The function σ and the vector s^λ in $\mathbf{s}=s^\lambda\partial\mathbf{f}_1/\partial\xi^\lambda$ satisfy the scalar equations $\mathbf{N}_2\cdot\partial\mathbf{f}_2/\partial\xi^\lambda=0$. The authors compute only the case in which $\mathbf{f}_1$ is a developable. *V. Hlavatý* (Prague).

Haimovici, Adolf. Sur une certaine déformation des congruences de sphères. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] **1**, 238–255 (1946).

Ce travail envisage une généralisation, dans l'espace conforme, des correspondances ponctuelles entre deux surfaces de l'espace ordinaire conservant l'angle des plans tangents en un couple quelconque de points infiniment voisins. Les plans tangents aux différents points d'un surface sont remplacés par des sphères tangentes, et il s'agit d'étudier les correspondances entre deux surfaces telles que l'angle des sphères tangentes en deux points infiniment voisins de l'une des deux surfaces soit égal à l'angle des sphères tangentes aux points infiniment voisins correspondants sur l'autre surface. Faisant usage du repère pentasphérique d'É. Cartan [Bull. Soc. Math. France **45**, 57–121 (1917)], l'auteur commence par établir, pour les correspondances de l'espèce indiquée, un théorème analogue au théorème de Péterson dans l'espace ordinaire. Il définit, sur une surface (S) enveloppe d'une congruence de sphères (Γ), deux directions conjuguées issues d'un même point M par la condition que l'une d'elles est tangente au cercle commun à la sphère de (Γ) relative à M et à la sphère infiniment voisine relative au point infiniment voisin de M sur l'autre direction. Il montre ensuite que, dans une correspondance du type étudié, il existe un réseau conjugué de l'une des deux surfaces ayant pour homologue un réseau conjugué de l'autre surface. Le problème de la recherche des congruences (Γ) admettant une déformation en congruences $(\bar{\Gamma})$ avec conservation simultanée de l'angle de deux sphères infiniment voisines homologues et des angles de deux couples de directions homologues sur deux nappes homologues des enveloppes des deux congruences, est mis en équations et discuté dans quelques cas particuliers. Suit la recherche des congruences (Γ) déformables en congruences $(\bar{\Gamma})$ (toujours avec conservation de l'angle de deux sphères infiniment voisines), de manière que le réseau de Péterson conjugué commun à deux nappes homologues des enveloppes de (Γ) et $(\bar{\Gamma})$ soit formé de lignes de courbure, ces lignes se correspondant à la fois sur les deux nappes de l'enveloppe de (Γ) et de $(\bar{\Gamma})$. Il s'agit, en somme, d'une transformation de congruences de sphères de Ribaucour. Le cas particulier où les nappes homologues de (Γ) et $(\bar{\Gamma})$ sont cerclées est examiné plus spécialement, et il est montré que la généralité des nappes focales cerclées qui interviennent dans ce cas est celle de deux fonctions arbitraires d'une variable. *P. Vincensini* (Besançon).

Haimovici, Adolf. Sur une correspondance entre une congruence de droites et une congruence de sphères. Disquisit. Math. Phys. **5**, 99–113 (1946).

Dans ce travail une congruence de sphères et une congruence de droites sont associées de façon que les deux points caractéristiques d'une sphère de la première congruence soient situés sur la droite correspondante de la deuxième. A toute congruence de sphères on peut évidemment associer une congruence de droites pour qu'il en soit ainsi, et l'objet principal de l'article est la recherche des congruences de sphères (Σ) que l'on peut associer à une congruence de droites donnée (Γ). Il existe, pour une congruence (Γ) donnée, une infinité de congruences (Σ) dépendant de deux fonctions arbitraires d'une variable et de quatre constantes arbitraires. L'auteur étudie, en particu-

lier, les cas où une nappe focale de la congruence de sphères (Σ) se confond avec une nappe focale de la congruence de droites associée (Γ): cela n'a lieu que lorsque la nappe focale de (Γ) se réduit à une courbe. Il envisage aussi les conditions que doit remplir (Γ) pour que parmi les (Σ) associées figurent des congruences de sphères de courbure d'une surface, et termine par l'étude du cas où la congruence (Γ) est parabolique. [Dans un mémoire antérieur [Ann. Sci. École Norm. Sup. (3) **61**, 119–147 (1944); ces Rev. **7**, 262], l'auteur du présent compte rendu avait déjà étudié le problème de la recherche des congruences de sphères (Σ) admettant pour congruence des cordes de contact une congruence arbitrairement donnée (Γ), et avait déterminé le degré de généralité de sa solution en se plaçant à un point de vue différent de celui de l'article actuel.] *P. Vincensini* (Besançon).

Haimovici, M. Sur la géométrie d'une famille de ∞^1 congruences de courbes dans un plan. Disquisit. Math. Phys. **5**, 75–98 (1946).

On sait qu'à une équation différentielle du deuxième ordre (A) $y''=f(x, y, y')$ on peut associer d'une manière intrinsèque un espace d'éléments linéaires à connexion projective, qui fournit les invariants de l'équation (A). Si on se donne une intégrale première de (A),

$$\text{(B)} \qquad \varphi(x, y, y', z)=0,$$

où z est une constante d'intégration, cette intégrale première définit une famille de ∞^1 congruences de courbes du plan. L'auteur applique la méthode du repère mobile d'É. Cartan à la détermination d'un espace d'éléments linéaires à connexion affine sans torsion, intrinsèquement lié à (B). Les géodésiques de cet espace sont les courbes intégrales de (B). Le mémoire contient la détermination complete des tenseurs de courbure de l'espace, la forme des identités de Bianchi, etc. *A. Lichnerowicz* (Strasbourg).

***Hachtroudi, M. Les Espaces Normaux. I. Les Espaces d'Éléments à Connexion Affine Normale. II. Les Espaces d'Éléments Linéaires à Connexion Weylienne Normale.** Université de Téhéran, Faculté des Sciences, 1945. v+83 pp.

L'auteur recherche les invariants d'une équation différentielle du second ordre relativement soit au groupe des transformations ponctuelles à jacobien unitaire, soit au groupe des transformations ponctuelles conformes du plan. A cet effet, il attache à l'équation, selon la méthode d'É. Cartan, dans le premier cas un espace à deux dimensions d'éléments linéaires à connexion affine normale et dans le second un espace à deux dimensions d'éléments linéaires à connexion de Weyl. Le premier problème est ensuite généralisé au cas d'un nombre quelconque de dimensions et les espaces à connexion affine normale correspondants étudiés. Le second problème traité (qui n'a évidemment de sens que pour 2 dimensions) apparait comme une extension de celui étudié par E. Kasner et J. De Cicco [Bull. Amer. Math. Soc. **49**, 236–245 (1943); ces Rev. **4**, 226]. *A. Lichnerowicz.*

***Pimiä, Lauri. Die involutorischen Berührungstransformationen der höheren Kugelgeometrie.** C. R. Dixième Congrès Math. Scandinaves 1946, pp. 245–248. Jul. Gjellerups Forlag, Copenhagen, 1947.

Cf. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. nos. **4** (1941); **16**, **21** (1943); **32** (1945); these Rev. **7**, 483; **8**, 350.

Castoldi, L. Sulla curvatura media di una varietà immersa in un'altra. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 928–934 (1946).

The author considers a set of hypersurfaces V_{m-1} immersed in a Riemannian space V_m which are geodesically parallel to a certain initial V^0_{m-1}. Let S be the volume of a small portion of V_{m-1}, corresponding to S^0 of V^0_{m-1}. The $y^1, \cdots, y^m$ denote a system of coordinates in V_m such that the geodesically parallel V_{m-1} are characterized by $y^m=\text{constant}$, the y^m denoting the arc of the geodesic normal to V^0_{m-1}. The author proves that, if Ω denotes the mean curvature of V^0_{m-1}, then

$$\Omega=-\lim_{S^0\to 0}(1/S^0)(\partial S/\partial y^m)_0.$$

The result is then generalized to an N_n immersed in a V_m. *E. T. Davies* (Southampton).

Castoldi, L. Caratterizzazione intrinseca degli spazi riemanniani in cui una funzione della distanza geodetica da un punto generico soddisfa l'equazione generalizzata di Laplace. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 1028–1034 (1946).

The author first introduces a system of coordinates in V_n for which $ds^2=(dx^1)^2+\sum_{r,s=2}^{n}A_{rs}dx^r dx^s$, where x^1 is the geodesic distance $S(P)$ from a certain origin P_0 to an arbitrary point P in the region of P_0. Having pointed out that this function $S(P)$ has intrinsic significance in the geometry of the V_n, he considers the problem of characterizing geometrically the spaces V_n for which there exists a function $f[S(P)]$ satisfying the generalized Laplace equation $\Delta f=g^{ij}f_{,ij}=0$. He proves that a necessary and sufficient condition for such a function to exist satisfying the generalized Laplace equation in a certain region of a point P_0 is that all the geodesic hypersurfaces of centre P_0 should have their mean curvatures constant. He also proves that there exists for every V_n a function satisfying the generalized Laplace equation at the point P_0 itself. *E. T. Davies.*

Castoldi, L. Sopra alcune proprietà caratteristiche delle V_n totalmente geodetiche rispetto ad una V_m ambiente. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 1064–1069 (1946).

Most of the results given in this note are known [see L. P. Eisenhart, Riemannian Geometry, Princeton University Press, 1926]. The final problem considered is the following. Given a field of unit vectors u^a which are normal to a totally geodesic V_n immersed in V_m, to determine the directions issuing from an arbitrary point of V_n along which the assigned vectors are parallel in V_m. The conclusion is that if $n>\frac{1}{2}(m-1)$ there are at least $2n-m$ such directions. *E. T. Davies* (Southampton).

Blum, Richard. Sur les identités de Bianchi et Veblen. C. R. Acad. Sci. Paris **224**, 889–890 (1947).

It is well known that, in a space with symmetric connection, one can introduce normal coordinates in such a way that at an arbitrary point P

$$(\Gamma^i_{ab})_P=0, \quad \left(\sum\frac{\partial\Gamma^i_{ab}}{\partial x^c}\right)_P=0, \quad \left(\sum\frac{\partial^2\Gamma^i_{ab}}{\partial x^c\partial x^d}\right)_P=0, \cdots.$$

From the way in which Vranceanu obtains these normal coordinates [in a forthcoming book, Leçons de Géométrie Différentielle], it follows that these are the only conditions the parameters Γ^i_{ab} are subject to in general. One can then determine the number of identities which the tensors $R^i_{abc,d}$

satisfy, R^i_{abc} being the curvature tensor. The Bianchi identities constitute a complete set of identities for the tensors $R^i_{abc,d}$ and so do the Veblen identities

$$R^i_{bac,d}+R^i_{cbd,a}+R^i_{dca,b}+R^i_{adb,c}=0.$$

A. Schwartz (State College, Pa.).

Chern, Shiing-shen. On the characteristic classes of Riemannian manifolds. Proc. Nat. Acad. Sci. U. S. A. 33, 78–82 (1947).

The object of the present note is to sketch the proof of a theorem determining the characteristic classes of the tangent bundle of a Riemannian manifold [see E. Stiefel, Comment. Math. Helv. 8, 305–343 (1936); H. Whitney, Lectures in Topology, pp. 101–141, University of Michigan Press, 1941; these Rev. 3, 133] in terms of the Riemannian metric. This generalizes a result of L. Pontrjagin [C. R. (Doklady) Acad. Sci. URSS (N.S.) 43, 91–94 (1944); these Rev. 6, 182], who proved the theorem in case the manifold was imbedded in a Euclidean space and the metric was the induced metric. The theorem in a certain sense generalizes the Gauss-Bonnet theorem, a result about 2-dimensional invariants of a 2-dimensional manifold, to r-dimensional invariants of an n-dimensional manifold.

H. Whitney.

Lichnerowicz, André. Sur une extension de la formule d'Allendoerfer-Weil à certaines variétés finslériennes. C. R. Acad. Sci. Paris 223, 12–14 (1946).

Die von Allendoerfer und A. Weil entdeckte, für $2p$-dimensionale Riemannsche Räume gültige Verallgemeinerung der Gauss-Bonnetschen Formel wird auf die Berwald-Cartanschen Räume übertragen; das sind diejenigen Finslerschen Räume, in denen die Winkelmetrik die konstante Krümmung 1 besitzt. Die Methode ist die des "repère mobile" von Cartan. Es wird ferner darauf hingewiesen, dass aus den Formeln dieser Note auch folgender Satz abgeleitet werden kann. Auf einer Sphäre gerader Dimension existiert keine reguläre Minkowskische Metrik, deren skalare Krümmung 0 ist.

H. Hopf (Zürich).

Lichnerowicz, André, et Thiry, Yves. Problèmes de calcul des variations liés à la dynamique classique et à la théorie unitaire du champ. C. R. Acad. Sci. Paris 224, 529–531 (1947).

The differential equations of the geodesics of the Finsler metric associated with the function $F(x^i, x'^i, x'^0)$ admit a first integral $\partial F/\partial x'^0=h$ and a relative integral invariant. Using these facts the authors prove that, given a function $L(x^i, x'^i, h)$ depending on an arbitrary parameter h, one and only one function $F(x^i, x'^i, x'^0)$ can be found whose geodesics give a parametric representation for the set of geodesics of L corresponding to the different values of h. When the metric defined by F is Riemannian, two particular cases are given, with their applications to classical dynamics, providing, in particular, the relation between the principle of Hamilton and that of Maupertuis.

E. T. Davies.

Bochner, S. Curvature in Hermitian metric. Bull. Amer. Math. Soc. 53, 179–195 (1947).

The author obtains properties of the local differential geometry furnished by a complex analytic n-dimensional positive-definite Hermitian metric $ds^2=g_{ij}dt_idt_j$ $(i, j=1, \cdots, 2n)$, where $(t_1, \cdots, t_{2n})$ are used to denote the conjugate complex numbers $(z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n)$; it is assumed that $g_{ij}=g_{ji}$, $g_{\alpha\beta}=g_{\alpha^*\beta^*}=0$, $g_{\alpha\beta^*}=\bar{g}_{\beta\alpha^*}$ $(\alpha,\beta=1, \cdots, n; \alpha^*=\alpha+n, \beta^*=\beta+n)$, so that $ds^2=2g_{\alpha\beta^*}dz_\alpha d\bar{z}_\beta$. The restriction is made that $\partial g_{\alpha\gamma^*}/\partial z_\beta=\partial g_{\beta\gamma^*}/\partial z_\alpha$ (nontorsion). If the curvature tensor R_{ijkl} is set up for g_{ij} by a formal generalization from the real case it is shown to determine uniquely the tensor g_{ij} (up to analytic coordinate transformations). Curvature (which is real) is defined with respect to a section determined by a pair of complex directions and is shown to be identically zero if at each point it is independent of section.

A holomorphic section is defined to be one determined by two complex vectors of which one is a nonreal multiple of the other; if the curvature at each point is independent of holomorphic section, it is shown to be constant for all points and holomorphic sections. Two spaces of the same constant holomorphic curvature b are equivalent and there is a coordinate system (z) for which $g_{\alpha\beta^*}=\delta_{\alpha\beta^*}/S-\frac{1}{2}b\bar{z}_\alpha z_\beta/S^2$, where $S=1+\frac{1}{2}b\sum z_\alpha\bar{z}_\alpha$. Such spaces are called Fubini spaces. If the Ricci (mean) curvature is at each point independent of direction, it is constant for all points and directions. In a Fubini space of holomorphic curvature b, the general sectional curvature lies between b and $\frac{1}{4}b$ and the Ricci curvature is $b(n+1)(2n-1)/(4n-4)$. If a Hermitian space is isometrically analytically imbeddable in a complex Euclidean space $ds^2=dw_1d\bar{w}_1+\cdots+dw_nd\bar{w}_n$, the curvature for every holomorphic section is nonpositive. Examples are given of Hermitian spaces not imbeddable in any such Euclidean space; in particular, ordinary hyperbolic space $ds^2=(1-z\bar{z})^{-2}dzd\bar{z}$ is such an example. Finally it is shown that a Hermitian metric invariant under a certain group of analytic homeomorphisms of a complex n-dimensional region has nonpositive curvature for holomorphic sections.

S. B. Myers (Ann Arbor, Mich.).

Wagner, V. Geometry of a space with an areal metric and its applications to the calculus of variations. Rec. Math. [Mat. Sbornik] N.S. 19(61), 341–406 (1946). (Russian. English summary)

This paper is concerned with the geometry arising in a calculus of variations problem for multiple integrals. If E_n is the Euclidean or Minkowski tangent space of a space X_n, an m-dimensional direction in E_n can be represented by a simple m-vector (Plücker tensor) $x^{\alpha_1\cdots\alpha_m}$. The equations expressing the condition that $x^{\alpha_1\cdots\alpha_m}$ is simple define a cone, the Grassmann cone in a Klein space of all m-vectors, the dimension of this space being $\binom{n}{m}$. The n-dimensional areal metric is defined in E_n by giving a simple m-vector $l^{\alpha_1\cdots\alpha_m}$ in E_n which defines a unit area in each E_m. The locus of all $l^{\alpha_1\cdots\alpha_m}$ corresponding to unit regions is an $m(n-m)$-dimensional surface of the Grassmann cone which is called the indicatrix of the m-dimensional areal metric in E_n and the study of the areal metric is thus reduced to the study of this indicatrix. A metric of this kind leads to a metric tensor g_{ij} of $m(n-m)$ dimensions but there are two affine connections. If $x^{\alpha_1\cdots\alpha_m}$ is a simple m-vector which is the base of an $(m+1)$-dimensional simplex with other sides $\underset{h}{x}^{\alpha_1\cdots\alpha_m}$ such that

$$x^{\alpha_1\cdots\alpha_m}=\sum_{h=1}^{m}\underset{h}{x}^{\alpha_1\cdots\alpha_m},$$

then the metric satisfies the minimal or maximal simplex axiom, respectively, if $\sigma\leqq\sum_{h=1}^m\underset{h}{\sigma}$ or $\sigma\geqq\sum_{h=1}^m\underset{h}{\sigma}$, where the σ's are the measures of the m-dimensional areas of the faces. A number of theorems are proved dealing with the necessary and sufficient conditions that an areal metric must fulfill in order to satisfy the minimal or maximal simplex axiom. By means of these ideas the author gives an inter-

pretation of the transversality conditions and obtains the Weierstrass necessary condition as well as the sufficient condition in the variational problem for multiple integrals.
M. S. Knebelman (Pullman, Wash.).

Davies, E. T. On metric spaces based on a vector density. Proc. London Math. Soc. (2) **49**, 241–259 (1947).

Finsler geometry and Cartan geometry can be treated by a general scheme which was developed by Schouten and Haantjes [Monatsh. Math. Phys. **43**, 161–176 (1936)]. The object of this paper is to make a study of two vectors in the general case, which play an essential rôle in further development. These vectors are: (1) the unit vector **l** associated with the element of support; (2) a vector **A**. Among the topics treated are: the absolute differential of the unit vector, angular metric, covariant derivation, Lie derivation, theory of hypersurfaces and curves, etc. *S. Chern.*

Debever, Robert. Les espaces métriques à quatre dimensions fondés sur la notion d'aire à deux dimensions. C. R. Acad. Sci. Paris **224**, 887–889 (1947).

The elements of the spaces are two-dimensional elements of contact, to each of which is associated a linear space in which there are defined from the metric an element of volume and a quadratic family of biplanes. The associated linear space is Euclidean if the family consists of biplanes tangent to a quadric cone. If this is the case, the space can be regarded as a locally Euclidean space of two-dimensional elements of contact. *S. Chern* (Shanghai).

Debever, Robert. Sur une classe de formes quadratiques extérieures et la géométrie fondée sur la notion d'aire. C. R. Acad. Sci. Paris **224**, 1269–1271 (1947).

The problem of equivalence for a two-dimensional element of area in an analytic variety V_4 of four dimensions is solved by means of Cartan's method of equivalence. From this solution it follows that a Euclidean connection can be defined in the space of two-dimensional elements of contact of V_4. This gives the analytic basis of some of the author's earlier results [see the preceding review]. It is also remarked that the problem of an m-dimensional element of area in an n-dimensional variety can be treated in a similar manner.
S. Chern (Shanghai).

Wang, Hsien-Chung. Path manifolds in a general space of paths. J. London Math. Soc. **21**, 134–139 (1946).

The following theorem is proved. In a general space of paths (in the sense of Douglas) of dimension n, a necessary and sufficient condition for every k-dimensional path-manifold to be twice differentiable for a certain k $(1<k<n)$ is that the coefficients of the projective connection of the space depend on position only. The theorem shows that differentiability assumptions do impose strong restrictions on the properties of the space. *S. Chern* (Shanghai).

Craig, Homer V. On the structure of intrinsic derivatives. Bull. Amer. Math. Soc. **53**, 332–342 (1947).

The author's purpose is to express the Mth order intrinsic derivative of a tensor T_{ab} (or T^{ab}, T^{ab}_{cd}) in terms of contracted extensors (contracted tensors of the extended point transformation) associated with T_{ab}. The procedure is the following: (1) a class of extensors E is constructed from T_{ab}; (2) by induction, the author shows that contracted combinations of the E's and the extended connection form the Mth order intrinsic derivatives. *N. Coburn.*

Viard, Jeannine. Calcul tensoriel gauche. C. R. Acad. Sci. Paris **224**, 543–545 (1947).

The author develops a tensor analysis with noncommutative components similar to the vector analysis of M. Cazin [same C. R. **222**, 992–994 (1946); these Rev. **8**, 121]. There are three types of two-index elements called vectorial tensors, true tensors and tensorial quantities. These may be covariant, contravariant or mixed. The product of a true tensor by a scalar operator need not be a true tensor and the sum of tensorial quantities need not be a tensorial quantity. Tensors may be symmetric or antisymmetric in one coordinate system and not in another. *O. Frink.*

NUMERICAL AND GRAPHICAL METHODS

Uhler, Horace S. Special values of $e^{k\pi}$, cosh $(k\pi)$ and sinh $(k\pi)$ to 136 figures. Proc. Nat. Acad. Sci. U. S. A. **33**, 34–41 (1947).

Values are given for $\exp(\pi/p)$, $\exp(-\pi/p)$, $\cosh(\pi/p)$ and $\sinh(\pi/p)$ for $p=2^n$ and $3\cdot 2^n$, $n=0, 1, \cdots, 6$, to 138 places, of which 136 are guaranteed as to accuracy. The values were computed from the author's radix table of logarithms to 137 places; independent checks were applied.
P. W. Ketchum (Urbana, Ill.).

***Lowan, A. N., Morse, P. M., Feshbach, H., and Lax, M. Scattering and Radiation from Circular Cylinders and Spheres. Tables of Amplitudes and Phase Angles.** U. S. Navy Department, Office of Research and Inventions, 1946. v+124 pp.

Let $J_n(x)$ and $Y_n(x)$ denote the Bessel functions of the first and second kinds, respectively, and let $j_n(x)=(\pi/2x)^{\frac{1}{2}}J_{n+\frac{1}{2}}(x)$, $y_n(x)=(\pi/2x)^{\frac{1}{2}}Y_{n+\frac{1}{2}}(x)$. Then, for $x=0.0(0.1)10.0$; $n=0(1)20$, the following functions are tabulated (in degrees, two-decimal accuracy; a prime denotes derivative with respect to x):

$$\begin{aligned}
\alpha_n(x)&=\arctan\{-xJ_n'(x)/J_n(x)\},\\
\beta_n(x)&=\arctan\{-xY_n'(x)/Y_n(x)\},\\
\delta_n(x)&=\arctan\{-J_n(x)/Y_n(x)\},\\
\delta_n^*(x)&=\arctan\{-J_n'(x)/Y_n'(x)\},\\
\gamma_n(x)&=\arctan\{\tan\delta_n(x)\cdot\sec\alpha_n(x)\cdot\cos\beta_n(x)\},
\end{aligned}$$

and their respective analogues obtained when J, Y are replaced by j, y. Moreover, for the same values of x and n, there are tables to six significant figures of the moduli of $J_n(x)+iY_n(x)$, $J_n'(x)+iY_n'(x)$, $j_n(x)+iy_n(x)$, $j_n'(x)+iy_n'(x)$. There are an introductory account of wave motion external to spheres and circular cylinders and a survey of the most important Bessel-function formulas, including approximate expressions. There is no reference to methods of computation or earlier tables, except Watson's in "A Treatise on the Theory of Bessel Functions" [Cambridge University Press, 1922]. [There is a misprint in the table of contents: in the definitions of tables 10 and 11, $n_n(x)$ and $n_n'(x)$ should be interchanged.] *C. J. Bouwkamp* (Eindhoven).

Olver, F. W. J. Note on a paper of H. Bateman. J. Appl. Phys. 17, 1127 (1946).

This paper gives values of $P_n(1-2e^{-t})$ correct to 15 decimals for $n=1(1)10$ and $t=1(1)20$. A similar table in a paper by Bateman [same J. 17, 91–102 (1946); these Rev. 7, 305] was not free from error. A correct table has now been given rather than a list of errors. *L. J. Comrie* (London).

Horton, C. W. Note on the zeros of $P_n^m(\cos\theta)$ and $dP_n^m(\cos\theta)/d\theta$ considered as functions of n. Bull. Amer. Math. Soc. 53, 153–155 (1947).

This gives tables of values of n for which $P_n^m(\cos\theta)$ and $dP_n^m(\cos\theta)/d\theta$ have zeros at $\theta=15°$, $30°$ or $45°$. Four to six values are given, to two decimals, for $m=0, 1, 2$ (96 values in all). The tables correct earlier ones given by B. Pal [Bull. Calcutta Math. Soc. 9, 85–95 (1917); 10, 187–194 (1919)] by cutting out certain values of n that are not zeros and by insertion of other zeros overlooked by Pal. The changes are all for small values of n, at the beginning of the tables. *J. C. P. Miller* (London).

Mikheladze, Š. On the subtabulation of mathematical tables. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoi SSR] 6, 397–405 (1945). (Georgian and Russian)

The author considers the problem of the m-fold subtabulation of a table of $f(x)$ originally given for $x=a+kh$ $(k=0, 1, 2, \cdots)$ to obtain a table for the arguments $x=a+kh_1$, where $h_1=h/m$. Formulas are given for forward, backward and central differences with remainders after r terms expressed in terms of the $(r+1)$th derivative of $f(x)$. *D. H. Lehmer* (Berkeley, Calif.).

Vaughan, Hubert. Some notes on interpolation. J. Inst. Actuar. 72, 482–497 (1946).

The author considers the problem of subtabulating at unit intervals a function for which values are given at intervals of k (an integer). He points out that any interpolation formula in which the interpolated value v_x is expressible in terms of the given values u_{mk} $(m=\cdots, -1, 0, 1, \cdots)$ in the form $v_x=\sum_{m=-\infty}^{\infty}L_{x-mk}u_{mk}$ is uniquely related to a smoothing formula which gives the smoothed values V_x of a sequence U_x by the relation $V_x=(1/k)\sum_{m=-\infty}^{\infty}L_{x-m}U_m$. He shows that if the coefficients satisfy the symmetry condition $L_{-y}=L_y$ the summation formula can be expressed in the form $V_x=[k]^n f(\delta)U_x$, where $[k]$ is the conventional summation operator $[k]w_x=\sum_{i=1}^{k}w_{x-\frac{1}{2}(k+1)+i}$, $n-1$ is the order of differences to which the given interpolation formula is correct, δ is the usual central difference operator and $f(t)$ is a series of nonnegative even powers of t. The ordinary case of interpolation by means of a polynomial of degree $n-1$ fitted to n consecutive centrally located given values is obtained by taking $f(t)$ as that polynomial of minimum degree which makes the smoothing formula correct to $(n-1)$th differences. He suggests judging the relative "smoothing power" of different interpolation formulas by comparing the values of the conventional "smoothing coefficient" S given, in this case, by $S^2=(1/20k^2)\sum_{y=-\infty}^{\infty}(\Delta^3L_y)^2$, for the corresponding smoothing formulas and discusses the problem of finding the interpolation formula of maximum "smoothing power" for a stated range and accuracy. Finally, he considers the situation in which it is desired to retain the given values without adjustment and to insert the intermediate values so as to satisfy the sole condition that the sum of the squares of the nth differences over the entire range of subtabulation shall be a minimum, and develops and illustrates a numerical method of obtaining the intermediate values. *T. N. E. Greville* (Washington, D. C.).

Beers, Henry S. Six-term formulas for routine actuarial interpolation. Record. Amer. Inst. Actuar. 34, 35–61 (1945).

This is the written discussion, customary in the Institute, of the paper in the same Record 33, 245–260 (1944); these Rev. 7, 84. Some new formulas are given by E. H. Wells and extensions and improvements are suggested by the author. *T. N. E. Greville* (Washington, D. C.).

Beers, Henry S. Modified-interpolation formulas that minimize fourth differences. Record. Amer. Inst. Actuar. 34, 14–20 (1945).

Continuing his earlier investigation [cf. the preceding review], the author derives the six-term interpolation formula for subdividing each interval into fifths with equidistant data which minimizes the sum of the squares of the fourth differences of the interpolated values, subject to the requirement that the formula is correct to third differences but permitting a fourth difference adjustment in the given values, and assuming the fourth differences of the given values to be independent random variables with zero mean and equal variance. *T. N. E. Greville.*

Beers, Henry S. Modified-interpolation formulas that minimize fourth differences. Record. Amer. Inst. Actuar. 34, 184–187 (1945).

This is the written discussion of the paper reviewed above.

Greville, T. N. E. Some extensions of Mr. Beers's method of interpolation. Record. Amer. Inst. Actuar. 34, 21–34 (1945).

The purpose of the method of interpolation under discussion is to secure from a set of observed values u_n, assumed subject to random errors, a new set of equally spaced subtabulated values v_{n+t}, where the v_{n+t} have been "smoothed" in the sense that the differences of v_{n+t} of a specified order have been minimized. For the actuarial applications under consideration, the values of t are taken at intervals of 0.2, that is, the subtabulation is for fifths of the given interval. The author presents seven tables of coefficients for formulas of the above type, ranging from a "three-term, minimized second-difference, ordinary interpolation formula correct to first differences" up to a "six-term, minimized fourth-difference, modified interpolation formula correct to third differences." *W. E. Milne* (Corvallis, Ore.).

Greville, T. N. E. Some extensions of Mr. Beers's method of interpolation. Record. Amer. Inst. Actuar. 34, 188–193 (1945).

This is the written discussion of the paper reviewed above.

Vernotte, Pierre. Les termes librement déterminés qui doivent compléter les développements limités. C. R. Acad. Sci. Paris 224, 32–34 (1947).

When a function $f(n)=\sum a_kf_k(n)$ is fitted to empirical data, and a few terms give an exact fit, it may be desirable to continue the series, extrapolating the a_k, particularly when $\sum_{0}^{\infty}f(n)$ is desired and no empirical values are available for $n>n_0$. *J. W. Tukey* (Princeton, N. J.).

Pinney, Edmund. Fitting curves with zero or infinite end points. Ann. Math. Statistics **18**, 127–131 (1947).

An expression is obtained, in terms of Jacobi polynomials, for the coefficients a_p in the function $f(x)=x^\alpha(1-x)^\beta\sum_p a_p x^p$, such that the moments $\mu_m=\int_0^1 x^m f(x)dx$ $(m=0, 1, \cdots, n)$ have preassigned values for given α, β. If α and β are not given, it is furthermore shown how they may be determined so that μ_{n+1} and μ_{n+2} have preassigned values.

T. N. E. Greville (Washington, D. C.).

✱Rasch, G. A principle for deriving the remainder terms in some serial expansions. C. R. Dixième Congrès Math. Scandinaves 1946, pp. 293–300. Jul. Gjellerups Forlag, Copenhagen, 1947.

The author's method of deriving the remainder terms in integral form for the classical formulae of interpolation, summation and mechanical quadratures is closely related to the method used by R. von Mises [J. Reine Angew. Math. **174**, 56–67 (1935)]. A reference older yet is to a paper by G. Peano [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) **22**₁, 562–569 (1913)]. *I. J. Schoenberg.*

Borghi, D. C. Sulle radici dell'equazione $\pi x=3\mathrm{tg}\,\pi x$. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **9(78)**, 41–46 (1945).

By means of Bessel functions the author derives, for the zeros of $\pi x=3\tan\pi x$, the formulas

$$x_0=1+(6/\pi)J_1(\tfrac{1}{3})L_0,\quad x_n=(-1)^n(6/\pi)J_{2n+1}(\tfrac{1}{3})L_n.$$

Here L_n is computed by iteration from

$$L_n=\cos[(2n+1)(6J_{2n+1}(\tfrac{1}{3})+\pi)\times\cos[(2n+1)(6J_{2n+1}(\tfrac{1}{3})+\pi)\cos[\cdots,$$

$L_0=0.940$, $L_n\sim 1$ $(n>0)$. The first approximate value for x_0 is 1.2991; to six decimals, $x_0=1.298115$. [It may be remarked that the closer approximate value $x_0=\tfrac{3}{2}-2/\pi^2=1.29736$ can be derived in an elementary way.] *S. C. van Veen.*

Cassina, Ugo. Su un nuovo metodo per la risoluzione numerica delle equazioni algebriche o trascendenti. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) **7(76)**, 35–61 (1943).

A modified version of Newton's method. Let f be differentiable on $[x, x_0]$ and vanish there only at x. If $0<d\leqq|f'(y)|\leqq D<\infty$ for $x\leqq y\leqq x_0$, and

$$x_{n+1}=x_n-(f(x_n)/D),$$

then the intervals $[y_n, x_n]$ shrink down on x, where

$$y_{n+1}=x_n-(f(x_n)/d).$$

Extensions to cases where (i) the derivative is replaced by the four derivative numbers; (ii) only the sequence x_n is considered. [The author states the theorems so that (y_n, x_n) contains x, which requires slightly stronger hypotheses.]

If φ is continuous on $[x, x_0]$, and if L' and L_1 are the supremum and infimum of the absolute values of the derivative numbers of φ on this interval, then $L_1<1$ implies that $x_{n+1}=\varphi(x_n)$ is a convergent process and this implies that $L'<1$. [This requires an additional hypothesis to ensure that $\varphi(x_0)$ belongs to $[x, x_0]$.] *J. W. Tukey.*

Leppert, E. L., Jr. An application of IBM machines to the solution of the flutter determinant. J. Aeronaut. Sci. **14**, 171–174 (1947).

The flutter equation is obtained in the form of a determinant set equal to zero, with complex numbers as elements and with the unknown occurring linearly in the first element of the diagonal. This determinant is reduced successively to lower orders by the introduction of zeros down the last column. The reduction is performed by IBM machines, including a multiplier, in such a way as to avoid divisions until the final step. *P. W. Ketchum* (Urbana, Ill.).

Lévy, Paul. Expression asymptotique de la longueur de l'ellipse infiniment aplatie. C. R. Acad. Sci. Paris **224**, 24–25 (1947).

The classical expression for the circumference of an ellipse (semi-axes a and $b=a(1-e^2)^{\frac{1}{2}}$), $4l=2\pi aF(\tfrac{1}{2}, -\tfrac{1}{2}; 1; e^2)$, is unsuitable for computation if $b/a\to 0$ $(e\to 1)$. The author proves that $\lim_{b\to 0}(l-a)/(b^2/2a)\log(a/b)=1$.

[The reviewer has derived many exact expressions for the same quantity. The simplest is

$$l=a\left\{F\left(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; \frac{b^2}{a^2}\right)+\frac{b^2}{2a^2}\log\frac{4a}{b}\cdot F\left(\tfrac{1}{2}, \tfrac{3}{2}; 2; \frac{b^2}{a^2}\right)\right.$$
$$\left.-\frac{2}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)(2m+2)}\sum_{q=m}^{\infty}\frac{\Gamma(q+\frac{1}{2})\Gamma(q+\frac{3}{2})}{q!(q+1)!}\left(\frac{b}{a}\right)^{2q+2}\right\}.$$

See Nederl. Akad. Wetensch., Proc. **44**, 1198–1203 (1941), formula (47); these Rev. **7**, 338.] *S. C. van Veen.*

✱Hartree, D. R. Calculating Machines. Recent and Prospective Developments and Their Impact on Mathematical Physics. Cambridge, at the University Press; New York, The Macmillan Company, 1946. 40 pp. (2 plates). \$.75.

The author reports his experiences while operating the ENIAC to obtain solutions of certain nonlinear differential equations and gives his ideas as to the influence of fast automatic calculating machines on the future of applied mathematics. As an illustration he cites the numerical solution of a particular nonlinear ordinary differential equation which involved operations with 200,000 digits. Merely to write down these digits by hand, without any computing, would require two eight hour days, but the ENIAC did the entire job in four minutes. A general account is given of the differences between analogue and digital machines, with an enumeration of the components of the latter, such as units for arithmetical operations, memory, transfer, input, output, master control, etc., followed by a somewhat more detailed account of the ENIAC itself. The fact that in the programming one must anticipate all possibilities, e.g., division by zero, and give the machine appropriate instructions for such contingencies, is stressed.

The availability of machines such as the ENIAC, and others of more advanced types with more adequate memories (the ENIAC stores only 20 numbers, not counting "slow" storage in the form of punched cards) and more automatic control features, such as the Harvard automatic sequence controlled calculator, capable of performing multiplications at the rate of a million an hour, should open up to the mathematical physicist a whole range of problems hitherto considered insoluble. In the case of two-dimensional problems governed by nonlinear partial differential equations, it should be entirely practicable to cover the given region by a reasonably fine network of points, replace the derivatives by finite differences, and obtain a solution by a step-by-step or iterative procedure. The impact of large scale computing equipment should also make itself felt on theoretical investigations by changing our viewpoint as to

The Harvard Machine, although it has desirable automatic control features not possessed by the ENIAC, is much slower; the time required to multiply two ten-digit numbers on the ENIAC is only about three milliseconds as (next page)
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contrasted with about three seconds for the same multiplication on the Harvard Machine.

what constitutes a simplification of the problem. For instance, the author considers that minimizing a quadratic form may be a more practical means of solving a large system of linear equations than the classical procedures. Iterative procedures will receive increased attention. Integral equations are likely to be preferred over differential equations. *P. W. Ketchum* (Urbana, Ill.).

***Hartree, D. R. The application of the differential analyser to the evaluation of solutions of partial differential equations.** Proc. First Canadian Math. Congress, Montreal, 1945, pp. 327–337. University of Toronto Press, Toronto, 1946. \$3.25.

Minorsky, N. A dynamical analogue. J. Franklin Inst. **243**, 131–149 (1947).

This paper outlines certain possibilities for graphical integration of ordinary differential equations by means of a dynamical analogue of a pendulum type. If a circular coil is free to rotate through a small angle θ about its diameter, the couple M acting on the movable system is of the form $M=\lambda i_1i_2 \sin\theta \doteq \lambda i_1i_2\theta$, where λ is a constant and i_1, i_2 are currents in solenoid and coil. The equation of motion of the system is then, for small θ, $I\ddot{\theta}+(\lambda i_1i_2)\theta=0$. The paper discusses various means by which the currents i_1 or i_2 or both may be made to vary in a prescribed way [in particular, by means of photo-integraphs; see T. S. Gray, same J. **212**, 77–102 (1931)] either as functions of t or in a manner determined by the equation itself, i.e., as functions of θ, of $\dot{\theta}$ or of both. Typical equations that may be solved by means of this analogue are

$$\ddot{\theta}+F(t)\psi(\theta)=0,$$
$$\ddot{\theta}-\mu\{a^2-g(\theta)\}\dot{\theta}+\theta=0,$$
$$\ddot{\theta}+\{q(t)+\lambda p(t)\}\theta=0.$$

In this case two coils (in two solenoids) are rigidly connected.

Equations of similar types, but with a "forcing term" independent of θ and of its derivatives, differential equations of higher orders (reduced to a set of two or more simultaneous equations, i.e., equations in which, for example, the term $\dot{\theta}(x-h)$ occurs, with a "time-lag" h) are also briefly considered. *J. C. P. Miller* (London).

Wilner, I. A. Analytical functions of a complex variable of the first nomographic class and their nomograms. C. R. (Doklady) Acad. Sci. URSS (N.S.) **53**, 187–190 (1946).

A functional relationship $F(w, z)=0$, where F is analytic and $w=p_1+ip_2$, $z=a+ib$, is defined to belong to the first nomographic class if it can be reduced to two real equations of the canonical form $f(p_i)X(a)+g(p_i)Y(b)+h(p_i)=0$, for $i=1, 2$. For such functions there exist nomographs with straight scales for a and b. The author gives a determination of all such functions in terms of elementary functions and elliptic integrals. The canonical representations of these elliptic integrals are also given. *P. W. Ketchum.*

Lordi, Luigi. Sulla decomposizione del premio nell'assicurazione vita. Rend. Accad. Sci. Fis. Mat. Napoli (4) **9**, 92–98 (1939).

ASTRONOMY

Wintner, Aurel. On a precession formula. Ann. Chaire Phys. Math. Kiev **4**, 169–183 (1939). (Ukrainian and English)

Fatou's asymptotic formula [Acta Astronomica. Ser. **2**, 101–164 (1931)] for the periods of certain integrable motions contains mistakes revealed when the present author applied it to questions connected with the precession of perihelion. The correct result is obtained in the present paper. The Lagrangian function for a conservative system with one degree of freedom is taken to be $L=a(q)q'^2+b(q)q'+c(q)$. The energy integral for this is $q'^2=f(q;h)=\{c(q)+h\}/a(q)$ (h constant). It is known that a nontrivial solution $q=q(t)$, bounded as $t\to\infty$, exists if the equation $f(q;h)=0$ has two simple roots $\alpha(h)$, $\beta(h)$ such that $f(q;h)\neq 0\,(\alpha<q<\beta)$, and $\alpha<q(t)<\beta$ for at least one t. Under these conditions there is a primitive period

$$T=T(h)=2\int_{\alpha(h)}^{\beta(h)}\frac{dq}{\sqrt{f(q;h)}}.$$

From this is deduced the asymptotic formula

$$T=2\pi\{-\tfrac{1}{2}f^{\mathrm{II}}(\gamma;h)\}^{-\frac{1}{2}}\{1+\tfrac{1}{16}\epsilon^2(A^2-3B)+o(\epsilon^2)\}$$

(ϵ small), where

$$\gamma=\tfrac{1}{2}(\beta+\alpha),\quad \epsilon=\tfrac{1}{2}(\beta-\alpha),\quad A=f^{\mathrm{III}}(\gamma;h)/f^{\mathrm{II}}(\gamma;h),$$
$$B=f^{\mathrm{IV}}(\gamma;h)/f^{\mathrm{II}}(\gamma;h)$$

and Roman indices denote partial derivatives with respect to q. The formula is applicable to Bertrand's problem of closed orbits and to the deduction of the Einstein shift. [The paper contains a misprint in formula (18), which gives $c=A^2\cdot 3B$ instead of A^2-3B.] *H. S. Ruse* (Leeds).

Nobile, Vittorio. Il termine solare dell'aberrazione e la struttura del sistema planetario. Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **12**, 25–49 (1942).

L'auteur a posé le problème d'évaluer la correction d'aberration de la lumière due au mouvement du soleil autour du centre de masse G du système solaire tout entier (terme solaire d'aberration). Il a démontré que le soleil S se meut uniformément sur une circonférence de centre G et il a indiqué la possibilité de calculer le rayon et la vitesse angulaire de ce mouvement au moyen d'observations stellaires sur l'aberration de la lumière. Il a remarqué encore que ses recherches peuvent conduire à la détermination de la masse totale du système solaire et de la position du centre de masse de l'ensemble des planètes ultra-plutoniennes dont il admet l'existence. [Voir Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **29**, 656–664 (1939).] G. Silva a fait remarquer que la position théorique du problème n'est pas rigoureuse parce que Nobile néglige dans ses calculs des quantités dont l'ordre de grandeur est le même des termes qu'il a pour but d'évaluer. Dans ce mémoire, l'auteur repond aux objections de Silva. *G. Lampariello* (Messina).

Silva, Giovanni. Il termine solare dell'aberrazione e la struttura del sistema planetario. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) **2**, 967–975 (1941).

Cette note est la réponse de l'auteur au travail de Nobile analysé ci dessus. Silva y expose les raisons pour lesquelles il pense que l'analyse de Nobile n'est pas suffisante pour atteindre le but envisagé. *G. Lampariello.*

Casale, Ambrogio. Generalizzazione di un lemma di Halphen. Ist. Lombardo Sci. Lett. Cl. Sci. Mat. Nat. Rend. (3) 9(78), 9–14 (1945).

The following theorem is established: if a particle P, subject to a positional force and to an arbitrary tangential force (which may depend on velocity and time), describes a plane orbit whenever its initial velocity is parallel to one of two given planes, then the positional force is central or has constant direction. This generalizes a theorem of Halphen [C. R. Acad. Sci. Paris 84, 939–941 (1877)].

W. Kaplan (Ann Arbor, Mich.).

García, Godofredo. The generalization of the equality of Lagrange and the inequality of Sundman in the case of more than three bodies. Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadística 1, 129–136 = Bol. Fac. Ingen. Montevideo 3 (Año 10), 177–184 (1946). (Spanish)

This is essentially the same as a previous paper [Proc. Nat. Acad. Sci. U. S. A. 28, 425–427 (1942); these Rev. 4, 57]. *W. Kaplan* (Ann Arbor, Mich.).

Agostinelli, Cataldo. Sull'area delle orbite cometarie. Atti Soc. Nat. Mat. Modena (6) 76, 64–67 (1945).

Following a suggestion by Pierucci that each comet arose from a planet or an asteroid having the same orbital area, the author shows how a planet could let loose a fragment whose subsequent orbit would have arbitrary eccentricity but the same area as that of the planet. *W. Kaplan.*

Armellini, Giuseppe. Sopra l'origine dei pianeti dal sole. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. (7) 4, 406–410 (1943).

It is suggested that the planets may have arisen from the sun simply because of an original excessive angular velocity of the sun, so that the attractive force of the sun could not balance the centripetal force. It is shown that such an assumption leads to a value of the moment of inertia of the sun which is consistent with the present-day theory of distribution of mass in the sun. *W. Kaplan.*

Armellini, G. Sul secondo problema fondamentale della cosmogonia. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 486–493 (1946).

The author proposes an explanation of the typical ellipsoidal structure of nebulae on the basis of a condensation process in a large spherical mass of gas of low density rotating about its axis. It is shown how the condensing particles would be subjected to forces tending to concentrate them in an equatorial plane in which they would move in approximately circular orbits. *W. Kaplan.*

Pignedoli, Antonio. Sulla stabilità delle configurazioni ellissoidali di una massa continua disgregata e stratificata soggetta alla propria gravitazione. Atti Soc. Nat. Mat. Modena (6) 75, 165–172 (1944).

In a previous paper [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 79, 332–345 (1944); these Rev. 7, 494] the author analyzed the motion of a continuous medium subject to internal gravitational forces and determined solutions such that the system is ellipsoidal and rotates as a rigid body about a fixed axis. It is now shown that these solutions are unstable. *W. Kaplan* (Ann Arbor, Mich.).

Mineo, C. Su una condizione necessaria per la stratificazione d'un astro fluido rotante in equilibrio relativo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 21–27 (1946).

The author considers the classical problem of figures of equilibrium of a perfect fluid. He demonstrates that a rotating heterogeneous body cannot have homothetic surfaces as surfaces of equal density. He criticises a proof of a similar theorem given by Wavre [Figures Planétaires et Géodésie, Gauthier-Villars, Paris, 1932, pp. 52–59].

E. J. Moulton (Evanston, Ill.).

Mineo, M. Superficie d'equilibrio terrestri chiuse. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 1, 1073–1078 (1946).

The author reviews the results of Pizetti [same Rend. (5) 10_2, 35–39 (1901)] concerning the existence of closed terrestrial surfaces of equilibrium. The analysis of the internal problem is less satisfying than that of the external problem, for the former is based on hypotheses concerning the density which necessarily leave the conclusions physically problematical. *E. F. Beckenbach* (Los Angeles, Calif.).

Woolley, R. v. d. R. Stellar opacity. Monthly Not. Roy. Astr. Soc. 106, 108–112 (1946).

The author attempts to base a theory of the stellar absorption coefficient on empirical facts about X-ray spectra. He derives this idea from a remark by Eddington in "The Internal Constitution of the Stars" [Cambridge University Press, 1926]. He expresses the empirical laws for the absorption coefficient for K-electrons and L-electrons in terms of oscillator strengths. He assumes a law for the distribution of oscillators of different critical frequencies. Then, taking into account the ionization of the different oscillators by means of the usual ionization equation, he calculates the Rosseland mean of the absorption caused by these oscillators. For the distribution of oscillators over critical frequencies he assumes a law which is a power n of the frequency. With this assumption he arrives at the conclusion that the opacity is proportional to $\rho T^{-(3+n)}$ in giant stars and to T^{-n} in dwarf stars. The case $n=0$ is suggested as a good approximation for the actual distribution of oscillators.

G. Randers (Oslo).

✶Qvist, Bertil. L'intégration d'un modèle d'étoile. C. R. Dixième Congrès Math. Scandinaves 1946, pp. 363–375. Jul. Gjellerups Forlag, Copenhagen, 1947.

The model considered is purely radiative, with constant energy-generation ϵ and opacity κ given by Kramers' formula $\kappa=\kappa_0\rho T^{-7/2}$. For any mean molecular weight, there is a maximum mass; for a molecular weight 0.7, this is 280 times the sun's mass. Reasonable agreement with the observed mass-luminosity relation is got with a suitable choice of κ_0. *T. G. Cowling* (Bangor).

Odgers, G. J. Structure of sunspots. Monthly Not. Roy. Astr. Soc. 106, 101–107 (1946).

Radiative equilibrium is assumed everywhere. A simple solution of the radiative equations exhibits the main properties of a spot if the product of opacity κ and density ρ has one constant value below the umbra, a lower constant value below the penumbra and a still lower value below the rest of the disc. The different values of $\kappa\rho$ are explained in terms of changes in density and in numbers of H^- ions. Mechanical equilibrium is not considered. *T. G. Cowling.*

Walén, Claës. On the distribution of the solar general magnetic field and remarks concerning the geomagnetism and the solar rotation. Ark. Mat. Astr. Fys. **33A**, no. 18, 63 pp. (1947).

The stabilizing effect of rotation in the sun's core is supposed to keep convection parallel to the axis, save during periodic convulsions. Electromagnetic forces oppose any motion across the magnetic field and supply a "rigidity" to the material, because of which the convulsions are able to produce periodic changes in the internal angular velocity. In the outer layers, Nernst currents are important. The radial limitation of the field is explained in terms of turbulence; as gas rises, it expands and causes the lines of force to separate, whence an upward decrease in the field.

T. G. Cowling (Bangor).

RELATIVITY

Clark, G. L. The gravitational field of a rotating cohesive system. Proc. Cambridge Philos. Soc. **43**, 164–177 (1947).

The problem of a rotating cohesive system is treated by means of the linearized field equations. The gravitational potentials for a rotating sphere and a rotating nearly spherical surface are given but the paper contains no hint as to how they are obtained. An examination is made of the action of the field at great distances and a comparison is made with previous results obtained for the rotating rod.

M. Wyman (Edmonton, Alta.).

Clark, G. L. Note on the velocity of propagation of gravitation. Proc. Cambridge Philos. Soc. **43**, 178–182 (1947).

The author follows a suggestion made by Eddington that the speed of propagation of gravitation should be investigated by means of the world invariant $B^i_{jkl}B_i^{jkl}$. By investigating this invariant for a rotating system it is found that the speed of propagation of gravitation is the same as that of light. *M. Wyman* (Edmonton, Alta.).

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✱Järnefelt, G. On the one-body problem in the expanding universe. C. R. Dixième Congrès Math. Scandinaves 1946, pp. 160–171. Jul. Gjellerups Forlag, Copenhagen, 1947.

A special case of McVittie's line-element [Monthly Not. Roy. Astr. Soc. **93**, 325–339 (1933)] is considered, which represents the field of a mass particle at the origin of an Einstein-De Sitter universe. It is shown that the orbit of a test particle does not expand with the universe if the orbit is of planetary dimensions [cf. G. Järnefelt, Ann. Acad. Sci. Fennicae. Ser. A. **55**, no. 3 (1940); these Rev. 7, 341]. *A. E. Schild* (Princeton, N. J.).

Hu, N. Radiation damping in the general theory of relativity. Proc. Roy. Irish Acad. Sect. A. **51**, 87–111 (1947).

In recent papers by Einstein, Infeld and Hoffmann, on the relativistic theory of gravitation and motion, mass particles are treated as singularities of the field. Furthermore, it has been shown by them that the equations of motion of the bodies involved are direct consequences of the field. The present paper replaces the mass particles by finite distributions of matter and then solves the field equations by approximate methods much the same as those used in the papers mentioned above. The introduction of finite distributions of matter introduces the energy-momentum tensor and the field equations for empty space are replaced by those which hold inside and outside of matter. In considering the two-body problem suitable restrictions are made on the energy-momentum tensor and the field quantities are then expanded in terms of $\lambda = v/c$, where v is a characteristic velocity of the particles and c is the velocity of light. The author carries the approximation as far as terms of order λ^9 as it is this term that is believed to give the effect of radiation damping. *M. Wyman* (Edmonton, Alta.).

✱Wyman, Max. Isotropic solutions of Einstein's field equations. Proc. First Canadian Math. Congress, Montreal, 1945, pp. 90–93. University of Toronto Press, Toronto, 1946. \$3.25.

Nonstatic isotropic solutions of the relativistic field equations are investigated. The general line-element is shown to have one of four forms. This condition is necessary but not sufficient. *A. E. Schild* (Princeton, N. J.).

BIBLIOGRAPHICAL NOTES

on order

✱Parke, Nathan Grier, III. Guide to the Literature of Mathematics and Physics Including Related Works on Engineering Science. The McGraw-Hill Book Company, Inc., New York and London, 1947. xv+205 pp. \$5.00.

The first part of this book contains general information on the use of scientific literature and library facilities, encyclopaedias, reviewing journals, etc. The second and larger part consists of a subject index of mathematics and physics, containing for each topic a brief note for orientation and an annotated list of significant books. There are detailed indices. *R. P. Boas, Jr.* (Providence, R. I.).

Periodico di Matematiche (4) 25, no. 2 (1947).

This is a special number honoring the memory of F. Enriques. It contains, in addition to biographical material and a bibliography, Enriques' "Prefazioni agli Elementi di Euclide," prepared for an edition of Euclid which has not yet appeared.

✱Comptes Rendus du Premier Congrès Canadien de Mathématiques, Montréal, 1945. Proceedings of the First Canadian Mathematical Congress, Montreal, 1945. The University of Toronto Press, Toronto, 1946. xliv+367 pp. \$3.25.

This volume contains minutes of the Congress, three Discussions on pedagogical topics, a Symposium on statistics, six short research papers and nineteen lectures.

✱Symposium Dedicated to the Thirty Anniversary of the Scientific and Pedagogical Activity of Professor V. I. Romanovsky.

Issued in Tashkent in 1939, this volume contains a biography by A. N. Nikolaev, an account of Romanovsky's mathematical work by N. N. Nazarov, a list of Romanovsky's publications, and nos. 19–32 of Acta Univ. Asiae Mediae, Ser. V-a. Mathematica. The title is in Russian and English.